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In memory of Yuming's father, Zhenrong QIN
and mother, Xilan XIA

To Jianlin's parents, Huaijian ZHANG and Xiuyun CHEN

To Yuming's wife and son, Yu YIN and Jia QIN

To Jianlin's wife, son and daughter, Huijun SUN, Lenan ZHANG
and Yuxin ZHANG

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Foreword

It is well-known that the theory of models of fluid flow is a hot topic in the applied mathematics and physics. Over the past two decades, radiation hydrodynamics problems associated with radiation phenomena have been a challenge to many investigators. The theory of radiation hydrodynamics finds a wide range of applications, including such diverse astrophysical phenomena as waves and oscillations in stellar atmospheres and envelopes, nonlinear stellar pulsation, supernova explosions, stellar winds, and many others. It has also a direct application in other areas, for instance to the physics of laser fusion and reentry of vehicles.

This book is aimed at presenting some recent results on global well-posedness including asymptotic behavior of global solutions to some fluid models, such as combustion model of radiative gas, radiation hydrodynamics model, Navier–Stokes equations with capillary and p-th power Newtonian fluid model. These models have a similar structure, which consist of Navier–Stokes equations coupled with other equations or with other effects. Most of materials of this book are based on the research carried out by the authors and their collaborators in recent years. Some have been previously published only in original papers, and some have never been published until now.

This book is divided into six chapters. In chapter 1, we shall provide some basic concepts and inequalities in analysis for the readers' convenience. In chapter 2, we shall deal with global existence and exponential stability of spherically symmetric solutions to a compressible combustion radiative and reactive gas. Here our results improve the previous work [110, 112, 134, 135]. In chapter 3, we shall investigate the global existence, uniqueness and exponential stability of solutions to the one-dimensional Navier–Stokes equations with capillarity. In chapter 4, we are concerned with the exponential stability of solutions to the compressible p-th power Newtonian fluid with large initial data. In chapter 5, we shall deal with the global existence and asymptotic behavior of spherically symmetric solutions for the multi-dimensional infrarelativistic model. In chapter 6, we shall further concern a

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multi-dimensional infrarelativistic model and establish the global existence and asymptotic behavior of cylindrically symmetric solutions for the three-dimensional infrarelativistic model.

We sincerely wish that the reader will know the main ideas and the essence of basic theories and methods in establishing the global well-posedness, asymptotic behavior of solutions for the models considered in this book. Also, we hope that the reader can be stimulated by some ideas from this book to undertake further studies and research after having read the related references and bibliographic comments in this book.

We also want to take this opportunity to thank all the people who are concerned about us including our teachers, colleagues and collaborators.

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Chapter 1

Preliminary

In this chapter, we recall some basic results, most of which will be used in subsequent chapters. The reader can easily find detailed proofs in the related literature, see, *e.g.*, Adams [1], Bellman [4], Evans [38], Gronwall [45], Nirenberg [91–94], Pachpatte [95], Shen and Zheng [124] and Zheng [143].

1.1 Sobolev Spaces and Their Basic Properties

In this section, we shall present some basic concepts of Sobolev spaces and their properties.

1.1.1 Weak Derivatives and Their Properties

Suppose, as usual, that Ω is an open set in $\mathbb{R}^n$. We denote the space of continuous functions by $C(\Omega)$, the space of functions with continuous partial derivatives in Ω of order less than or equal to $k \in \mathbb{N}$ by $C^k(\Omega)$ and the space of functions with continuous derivatives of all orders by $C^{\infty}(\Omega)$.

If the set $\Omega \subset \mathbb{R}^n$ is non-empty, we denote by $\bar{\Omega}$ the closure of Ω in $\mathbb{R}^n$. We denote the support of a continuous function $f: \ \Omega \to \mathbb{R}^n$ by

$$\mathrm{spt} f := \overline{\{x \in \Omega | f(x) \neq 0\}}.$$

We denote by $C_c(\Omega)$ the space of continuous functions whose support is compactly contained in Ω, and by $C_c^{\infty}(\Omega)$ the space of functions with continuous derivatives of all orders and compact support in Ω. The completion of $C_c(\mathbb{R}^n)$ with respect to the uniform norm is the space $C_0(\mathbb{R}^n)$ of continuous functions that approach zero at infinity.

Definition 1.1.1. *A function* $f \in L^1_{\mathrm{loc}}(\Omega)$ *is weakly differentiable with respect to* x_i *if there exists a function* $g_i \in L^1_{\mathrm{loc}}(\Omega)$ *such that for all* $\phi \in C_c^{\infty}(\Omega)$,

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$$\int_{\Omega} f\partial_i\phi \, dx = -\int_{\Omega} g_i\phi \, dx.$$

Then function g_i is called the weak i-th partial derivative of f, and is denoted by $\partial_i f$.

Thus, for weak derivatives, the integration by parts formula

$$\int_{\Omega} f\partial_i\phi \, dx = -\int_{\Omega} \partial_i f\phi \, dx$$

holds by definition for all $\phi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^1_{\mathrm{loc}}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. The existence of a weak derivative is, however, not equivalent to the existence of a pointwise derivative almost everywhere.

Unless stated otherwise, we shall always interpret derivatives as weak derivatives, and we use the same notation for weak derivatives and continuous pointwise derivatives. Higher order weak derivatives are defined in a similar way.

Definition 1.1.2. *Suppose that* $\alpha \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *is a multi-index. A function* $f \in L^1_{\mathrm{loc}}(\Omega)$ *has weak derivative* $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ *if for all* $\phi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} (\partial^\alpha f)\phi \, dx = (-1)^{|\alpha|} \int_{\Omega} f(\partial^\alpha \phi) \, dx.$$

Although we will not make extensive use of the theory of distributions, it is useful to understand the interpretation of a weak derivative as a distributional derivative. In fact, the definition of the weak derivative by Sobolev, and others, is one motivation for the subsequent development of distribution theory by Schwartz.

Definition 1.1.3. *A sequence* $\{\phi_n : n \in \mathbb{N}\}$ *of functions* $\phi_n \in C_c^\infty(\Omega)$ *converges to* $\phi \in C_c^\infty(\Omega)$ *in the sense of test functions if*

(a) there exists $\Omega' \subset \Omega$ *such that* $\mathrm{spt}\phi_n \subset \Omega'$ *for every* $n \in \mathbb{N}$;
(b) $\partial^\alpha \phi_n \to \partial^\alpha \phi$ *as* $n \to +\infty$ *uniformly on* Ω *for every* $\alpha \in \mathbb{N}_0^n$.

The topological vector space $\mathcal{D}(\Omega)$ consists of $C_c^\infty(\Omega)$ equipped with the topology that corresponds to convergence in the sense of test functions.

A linear functional T is continuous if $\phi_n \to \phi$ in the sense of test functions implies that $< T, \phi_n > \, \to \, < T, \phi >$ in $\mathbb{R}$.

Definition 1.1.4. *A distribution on* Ω *is a continuous linear functional*

$$T : \mathcal{D}(\Omega) \to \mathbb{R}.$$

A sequence $\{T_n : n \in \mathbb{N}\}$ of distributions converges to T, written by $T_n \rightharpoonup T$, if $< T_n, \phi > \, \to \, < T, \phi >$ for every $\phi \in \mathcal{D}(\Omega)$. The topological vector space $\mathcal{D}'(\Omega)$ consists of distributions on Ω equipped with the topology corresponding to this notion of convergence. Thus, the space of distributions is the topological dual of the space of test functions.

One of the main advantages of distributions is that, in contrast to functions, every distribution is differentiable. The space of distributions may be thought of as the smallest extension of the space of continuous functions that is closed under differentiation.

Definition 1.1.5. *For* $1 \le i \le n$, *the i-th partial derivative of a distribution* $T \in \mathcal{D}'(\Omega)$ *is the distribution* $\partial_i T \in \mathcal{D}'(\Omega)$ *defined by*

$$< \partial_i T, \phi > \; = - < T, \partial_i \phi > \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

For $\alpha \in \mathbb{N}_0^n$, *the derivative* $\partial^\alpha T \in \mathcal{D}'(\Omega)$ *of order* $|\alpha|$ *is defined by*

$$< \partial^\alpha T, \phi > \; = (-1)^{|\alpha|} < T, \partial^\alpha \phi > \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Note that if $T \in \mathcal{D}'(\Omega)$, then it follows from the linearity and continuity of the derivative $\partial^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ on the space of test functions that $\partial^\alpha T$ is a continuous linear functional on $\mathcal{D}(\Omega)$. Thus, $\partial^\alpha T \in \mathcal{D}'(\Omega)$ for any $T \in \mathcal{D}'(\Omega)$. It also follows that the distributional derivative $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is linear and continuous on the space of distributions; in particular if $T_n \rightharpoonup T$, then $\partial^\alpha T_n \rightharpoonup \partial^\alpha T$.

Any function $f \in L^1_{\text{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$< T_f, \phi > \; = \int_\Omega f \phi \, dx.$$

Any distribution associated with a locally integrable function in this way is called a regular distribution.

A locally integrable function is weakly differentiable if its distributional derivative is regular, and its weak derivative is the locally integrable function corresponding to the distributional derivative.

The distributional derivative of a function exists even if the function is not weakly differentiable.

We collect here some useful properties of weak derivatives. The first result is a product rule.

Proposition 1.1.1. *If* $f \in L^1_{\text{loc}}(\Omega)$ *has weak partial derivative* $\partial_i f \in L^1_{\text{loc}}(\Omega)$ *and* $\psi \in C^\infty(\Omega)$, *then* ψf *is weakly differentiable with respect to* x_i *and*

$$\partial_i(\psi f) = (\partial_i \psi) f + \psi(\partial_i f).$$

The commutativity of weak derivatives follows immediately from the commutativity of derivatives applied to smooth functions.

Proposition 1.1.2. *Suppose that* $f \in L^1_{\text{loc}}(\Omega)$ *and that the weak derivative* $\partial^\alpha f$, $\partial^\beta f$ *exist for multi-indices* $\alpha, \beta \in \mathbb{N}_0^n$. *Then if any one of the weak derivatives* $\partial^{\alpha+\beta} f$, $\partial^\alpha \partial^\beta f$, $\partial^\beta \partial^\alpha f$ *exists, all three derivatives exist and are equal.*

The next result gives an alternative way to characterize weak derivatives as limits of derivatives of smooth functions.

Proposition 1.1.3. *A function* $f \in L^1_{\text{loc}}(\Omega)$ *is weakly differentiable in* Ω *if and only if there is a sequence* $\{f_n\}$ *of functions* $f_n \in C^\infty(\Omega)$ *such that* $f_n \to f$ *and* $\partial^\alpha f_n \to g$ *in* $L^1_{\text{loc}}(\Omega)$. *In that case the weak derivative of* f *is given by* $g = \partial^\alpha f \in L^1_{\text{loc}}(\Omega)$.

The following results are the weak versions of the product and chain rule.

Proposition 1.1.4. *(1) Suppose that* $a \in C^1(\Omega)$ *and* $u \in L^1_{\text{loc}}(\Omega)$ *is weakly differentiable. Then* au *is weakly differentiable and*

$$\partial_i(au) = a(\partial_i u) + (\partial_i a)u.$$

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function with* $f' \in L^\infty(\mathbb{R})$ *bounded, and* $u \in L^1_{\text{loc}}(\Omega)$ *is weakly differentiable. Then* $v = f \circ u$ *is weakly differentiable and*

$$\partial_i v = f'(u)\partial_i u.$$

(3) Suppose that $\phi\colon \Omega \to \tilde{\Omega}$ *is a* C^1*-diffeomorphism of* Ω *onto* $\tilde{\Omega} = \phi(\Omega) \subset \mathbb{R}^n$. *For* $u \in L^1_{\text{loc}}(\Omega)$, *define* $v \in L^1_{\text{loc}}(\tilde{\Omega})$ *by* $v = u \circ \phi^{-1}$. *Then* v *is weakly differentiable in* $\tilde{\Omega}$, *if and only if* u *is weakly differentiable in* Ω, *and*

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial \phi_j}{\partial x_i} \frac{\partial v}{\partial y_j} \circ \phi.$$

1.1.2 Sobolev Spaces

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 1.1.6. *Suppose that* Ω *is an open set in* $\mathbb{R}^n$, $k \in \mathbb{N}$, *and* $1 \leq p \leq +\infty$. *The Sobolev space* $W^{k,p}(\Omega)$ *consists of all locally integrable functions* $f : \Omega \to \mathbb{R}$ *such that*

$$\partial^\alpha f \in L^p(\Omega) \qquad \text{for all } 0 \leq |\alpha| \leq k.$$

Generally, we write $W^{k,2}(\Omega) = H^k(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \leq p < +\infty$ and

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \sup_\Omega |\partial^\alpha f|.$$

Meanwhile, $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. As usual, we identify functions that are equal almost everywhere. We will use these norms as the standard ones on $W^{k,p}(\Omega)$, but there are other equivalent norms, *e.g.*,

$$\|f\|_{W^{k,p}(\Omega)}=\sum_{|\alpha|\leq k}\left(\int_\Omega|\partial^\alpha f|^p\,dx\right)^{\frac{1}{p}},\quad \|f\|_{W^{k,p}(\Omega)}=\max_{|\alpha|\leq k}\left(\int_\Omega|\partial^\alpha f|^p\,dx\right)^{\frac{1}{p}}.$$

The space $H^k(\Omega)$ is a Hilbert space with the inner product

$$<f,g> \;=\sum_{|\alpha|\leq k}\int_\Omega(\partial^\alpha f)(\partial^\alpha g)\,dx.$$

1.1.3 Some Properties of Sobolev Spaces

To begin with, some functions defined on all of $\mathbb{R}^n$ can be approximated in the Sobolev norm by test functions.

Theorem 1.1.1. *For* $k\in\mathbb{N}$ *and* $1\leq p<+\infty$, *the space* $C_c^\infty(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$.

Now we define the Sobolev conjugate.

Definition 1.1.7. *If* $1\leq p<n$, *the Sobolev conjugate* p^* *of* p *is*

$$p^*=\frac{np}{n-p}.$$

Obviously,

$$\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}.$$

The next theorem is about the Sobolev inequalities. Before describing this inequalities and its proofs, we introduce some notation and give two preliminary inequalities.

For $1\leq i\leq n$ and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, let

$$x_i'=(x_1,\ldots,\hat{x}_i,\ldots,x_n)\in\mathbb{R}^{n-1},$$

where the "hat" means that the i-th coordinate is omitted. We can write $x=(x_i,x_i')$ and denote the partial derivative with respect to x_i by ∂_i.

For any smooth function f with compact support, the next inequality is the improvement of the fundamental theorem of calculus using the fact

$$\int_{-\infty}^{\infty}\partial_i f(t,x_i')\,dt=0.$$

Lemma 1.1.1. *Suppose that* $g : \mathbb{R} \to \mathbb{R}$ *is an integrable function with compact support such that* $\int_{-\infty}^{\infty} g(t)\,dt = 0$. *If*

$$f(x) = \int_{-\infty}^{x} g(t)\,dt,$$

then

$$|f(x)| \leq \frac{1}{2}\int_{-\infty}^{\infty} |g(t)|\,dt.$$

The following preliminary inequality can be estimated; the L^1-norm of a function of $x \in \mathbb{R}^n$ in terms of the L^{n-1}-norms of n functions of $x_i' \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

Lemma 1.1.2. *Suppose that* $n \geq 2$ *and* $\left\{g_i \in C_c^{\infty}(\mathbb{R}^{n-1}) : 1 \leq i \leq n\right\}$ *are nonnegative functions. Define* $g \in C_c^{\infty}(\mathbb{R}^n)$ *by*

$$g(x) = \prod_{i=1}^{n} g_i(x_i'),$$

then

$$\|g\|_{L^1} \leq \prod_{i=1}^{n} \|g_i\|_{L^{n-1}}.$$

This lemma can be proved by the induction and the Hölder inequality, thus we omit it.

Theorem 1.1.2. *Let* $1 \leq p < n$, *where* $n \geq 2$, *and let* p^* *be Sobolev conjugate of* p *given in definition 1.1.7. Then for all* $f \in C_c^{\infty}(\mathbb{R}^n)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Df\|_{L^p(\mathbb{R}^n)},$$

where

$$C = C(n,p) = \frac{(n-1)p}{2n(n-p)}.$$

Proof. First, we consider the case $p = 1$. For $1 \leq i \leq n$, we have

$$|f(x)| \leq \frac{1}{2}\int |\partial_i f(t, x_i')|\,dt.$$

Multiplying these inequalities and taking the $(n-1)$-th root, we have

$$|f|^{\frac{n}{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} g, \quad g = \prod_{i=1}^{n} \tilde{g}_i$$

where $\tilde{g}_i(x) = g_i(x_i')$ with

$$g_i(x_i') = \left(\int |\partial_i f(t, x_i')|\, dt\right)^{\frac{1}{n-1}}.$$

It follows from lemma 1.1.2 that

$$\int g dx \leq \prod_{i=1}^{n} \|g_i\|_{L^{n-1}}.$$

Thus we obtain

$$\int |f|^{\frac{n}{n-1}} dx \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\prod_{i=1}^{n} \int |\partial_i f|\, dx\right)^{\frac{1}{n-1}}.$$

Note that $\frac{n}{n-1} = 1^*$ is the Sobolev conjugate of 1.

Again noting that the arithmetic–geometric mean inequality,

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i,$$

we have

$$\int |f|^{\frac{n}{n-1}} dx \leq \left(\frac{1}{2n}\sum_{i=1}^{n} \int |\partial_i f|\, dx\right)^{\frac{n}{n-1}}$$

or

$$\|f\|_{L^{1^*}} \leq \frac{1}{2n}\|Df\|_{L^1}.$$

Next we consider the case $1 < p < n$. For any $s > 1$, we know

$$\frac{d}{dx}|x|^s = s\mathrm{sgn} x|x|^{s-1}$$

where "sgn" denotes the sign function. Thus,

$$\begin{aligned} |f(x)|^s &= \int_{-\infty}^{x_i} \partial_i |f(t, x_i')|^s\, dt \\ &= s\int_{-\infty}^{x_i} |f(t, x_i')|^{s-1}\mathrm{sgn}[f(t, x_i')]\partial_i f(t, x_i')\, dt, \end{aligned}$$

which, by lemma 1.1.1, implies

$$|f(x)|^s \leq \frac{s}{2}\int_{-\infty}^{\infty} |f^{s-1}(t, x_i')\partial_i f(t, x_i')|\, dt.$$

Multiplication of these inequalities gives

$$|f(x)|^{sn} \leq \left(\frac{s}{2}\right)^n \prod_{i=1}^{n} \int_{-\infty}^{\infty} |f^{s-1}(t, x_i')\partial_i f(t, x_i')|\, dt.$$

Let

$$g_i(x_i') = \left(\int_{-\infty}^{\infty} |f^{s-1}(t, x_i')\partial_i f(t, x_i')|\, dt\right)^{\frac{1}{n-1}}.$$

Applying lemma 1.1.2 and the Hölder inequality, we find

$$\begin{aligned} \|f\|_{L^{\frac{sn}{n-1}}}^{sn} &\leq \frac{s}{2} \prod_{i=1}^{n} \left\|f^{s-1}\partial_i f\right\|_{L^1} \\ &\leq \frac{s}{2} \prod_{i=1}^{n} \left\|f^{s-1}\right\|_{L^{p'}} \|\partial_i f\|_{L^p} \\ &= \frac{s}{2} \prod_{i=1}^{n} \|f\|_{L^{p'(s-1)}}^{s-1} \|\partial_i f\|_{L^p}. \end{aligned}$$

We choose $s > 1$ so that

$$p'(s-1) = \frac{sn}{n-1},$$

which holds if

$$s = \frac{p(n-1)}{n-p}, \qquad \frac{sn}{n-1} = p^*.$$

Then

$$\|f\|_{L^{p^*}} \leq \frac{s}{2} \left(\prod_{i=1}^{n} \|\partial_i f\|_{L^p}\right)^{\frac{1}{n}},$$

which, using the arithmetic–geometric mean inequality, leads to

$$\|f\|_{L^{p^*}} \leq \frac{s}{2n} \left(\sum_{i=1}^{n} \|\partial_i f\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Therefore, we complete the proof. □

Remark 1.1.1. *The Sobolev inequality in theorem 1.1.2 does not hold in the limiting case $p \longrightarrow n$, $p^* \longrightarrow +\infty$.*

Remark 1.1.2. *For $p = 1$, the best constant is*

$$C(n, 1) = \frac{1}{n\alpha_n^{\frac{1}{n}}}$$

where α_n is the volume of the unit ball, or

$$C(n, 1) = \frac{1}{n\sqrt{\pi}}\left(\Gamma\left(1 + \frac{n}{2}\right)\right)^{\frac{1}{n}}$$

where Γ is the gamma-function. For $1 < p < n$, the best constant is

$$C(n, p) = \frac{1}{n^{\frac{1}{p}}\sqrt{\pi}}\left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}}\left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(1+n-\frac{n}{p}\right)}\right)^{\frac{1}{n}}.$$

Let X, Y be two Banach spaces and $X \hookrightarrow Y$ denote that X is continuously embedded in Y.

Theorem 1.1.3. *Suppose that $1 \leq p < n$ and $p \leq q \leq p^*$ where p^* is the Sobolev conjugate of p. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and for all $f \in W^{1,p}(\mathbb{R}^n)$,*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}$$

with some constant $C = C(n, p, q) > 0$.

Remark 1.1.3. *Sobolev embedding gives a stronger conclusion for sets Ω with finite measure. In that case, $L^{p^*}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq p^*$, so $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*$, not just $p \leq q \leq p^*$.*

Definition 1.1.8. *A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ vanishes at infinity if for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^n : |f(x)| > \epsilon\}$ has finite Lebesgue measure.*

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < +\infty$, then f vanishes at infinity. Note that this does not imply that $\lim_{|x|\to+\infty} f(x) = 0$.

The Sobolev embedding theorem remains true for functions that vanish at infinity.

Theorem 1.1.4. *Suppose that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is weakly differentiable with $Df \in L^p(\mathbb{R}^n)$ where $1 \leq p < n$ and f vanishes at infinity. Then $f \in L^{p^*}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Df\|_{L^p(\mathbb{R}^n)}$$

where $C > 0$ is given in theorem 1.1.2.

As before, we prove this by approximating f with smooth compactly supported functions. We omit the details.

Therefore, if the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. Even though the function is weakly differentiable, it need not be continuous.

Theorem 1.1.5. *Let $n < p < +\infty$ and $\alpha = 1 - \frac{n}{p}$ with $\alpha = 1$ if $p = +\infty$. Then there are constants $C = C(n, p) > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$,*

$$[f]_\alpha \leq C\|Df\|_{L^p(\mathbb{R}^n)}, \tag{1.1.1}$$

$$\sup_{x\in\mathbb{R}^n} |f| \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}, \tag{1.1.2}$$

where $[\cdot]_\alpha$ denotes the Hölder seminorm $[\cdot]_{\alpha,\mathbb{R}^n}$.

Note that

$$\|f\|_{C^{0,\alpha}} = \sup |f| + [f]_\alpha$$

and denote by $C_0^{0,\alpha}(\mathbb{R}^n)$ the space of Hölder continuous functions f whose limit as $x \to +\infty$ is zero, meaning that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ if $x \in \mathbb{R}^n \backslash K$.

Theorem 1.1.6. *Let $n < p < +\infty$ and $\alpha = 1 - \frac{n}{p}$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$$

and there is a constant $C = C(n, p) > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}.$$

Theorem 1.1.7. *A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is Lipschitz continuous if and only if it is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

When $n < p \leq +\infty$, the above estimates can be used to prove that pointwise derivative of a Sobolev function exists almost everywhere and agrees with the weak derivative.

Theorem 1.1.8. *If $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ for some $n < p \leq +\infty$, then f is differentiable pointwise a.e. and the pointwise derivative coincides with the weak derivative.*

Therefore, if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise *a.e.* equivalent version of it) is continuous, and in fact Hölder continuous.

Suppose that Ω is a smooth, bounded domain in $\mathbb{R}^n$. We may cover the closure $\bar{\Omega}$ by a collection of open balls contained in Ω and open balls with center $x \in \partial\Omega$.

The statement of the embedding theorem for higher order derivatives extends in a straightforward way from the one for first order derivatives. For example,

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \qquad \text{if } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

The result for smooth bounded domains is summarized in the following theorem. Let $X \hookrightarrow Y$ denote a continuous embedding of X into Y, and $X \hookrightarrow\hookrightarrow Y$ denote a compact embedding.

Theorem 1.1.9. *Suppose that* Ω *is a bounded open set in* $\mathbb{R}^n$ *with* C^1 *boundary,* $k, m \in \mathbb{N}$ *with* $k \geq m$*, and* $1 \leq p < +\infty$*.*

(1) If $kp < n$*, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \text{for } 1 \leq q < \frac{np}{n-kp};$$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for } q = \frac{np}{n-kp}.$$

More generally, if $(k-m)p < n$*, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{m,q}(\Omega) \qquad \text{for } 1 \leq q < \frac{np}{n-(k-m)p};$$

$$W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega) \qquad \text{for } q = \frac{np}{n-(k-m)p}.$$

(2) If $kp = n$*, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \text{for } 1 \leq q < +\infty.$$

(3) If $kp > n$*, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\mu}(\bar{\Omega})$$

for $0 < \mu < k - \frac{n}{p}$ *if* $k - \frac{n}{p} < 1$*, for* $0 < \mu < 1$ *if* $k - \frac{n}{p} = 1$*, and for* $\mu = 1$ *if* $k - \frac{n}{p} > 1$*; and*

$$W^{k,p}(\Omega) \hookrightarrow C^{0,\mu}(\bar{\Omega})$$

for $\mu = k - \frac{n}{p}$ *if* $k - \frac{n}{p} < 1$*. More generally, if* $(k-m)p > n$*, then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{m,\mu}(\bar{\Omega})$$

for $0 < \mu < k - m - \frac{n}{p}$ *if* $k - m - \frac{n}{p} < 1$*, for* $0 < \mu < 1$ *if* $k - m - \frac{n}{p} = 1$*, and for* $\mu = 1$ *if* $k - m - \frac{n}{p} > 1$*; and*

$$W^{k,p}(\Omega) \hookrightarrow C^{m,\mu}(\bar{\Omega})$$

for $\mu = k - m - \frac{n}{p}$ *if* $k - m - \frac{n}{p} = 0$*.*

These results hold for arbitrary bounded open sets Ω if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.

1.2 Some Inequalities in Analysis

In this section, we shall recall some important inequalities in analysis. These inequalities furnish some powerful tools in establishing the global well-posedness and asymptotic behavior of solutions to nonlinear evolutionary differential equations arising from physics, fluid mechanics, and materials science, etc. More inequalities in analysis can be found in Qin [100–102].

1.2.1 The Young Inequality

The following result is a classical result called "the Young inequality".

Theorem 1.2.1. *Let f be a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then*

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab \tag{1.2.1}$$

with f^{-1} is the inverse function of f. Equality holds in (1.2.1) if and only if $b = f(a)$.

If we take $f(x) = x^{p-1}$ with $p > 1$ in the above theorem, then we have the next corollary.

Corollary 1.2.1. *Let $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $p > 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.2.2}$$

If $0 < p < 1$, then

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.2.3}$$

The equalities in (1.2.2) and (1.2.3) hold if and only if $b = a^p$.

In corollary 1.2.1, if we replace a and b with ϵa and $\epsilon^{-1} b$, respectively, we can get the next corollary.

Corollary 1.2.2. *Let $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $p > 1$, then for any $\epsilon > 0$,*

$$ab \leq \frac{\epsilon^p a^p}{p} + \frac{b^q}{q\epsilon^q}. \tag{1.2.4}$$

In addition, the Young inequality has several variants in the following.

Corollary 1.2.3. *Let* a, $b \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *with* $1 < p < +\infty$. *Then*

(1) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$;
(2) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\epsilon^{\frac{1}{q}}} + \frac{b\epsilon^{\frac{1}{p}}}{q}$;
(3) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b$, $0 < \alpha < 1$.

1.2.2 The Poincaré Inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 1.2.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. There exists a positive constant* $C = C(n, p)$ *such that for every* $u \in W_0^{k,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}. \tag{1.2.5}$$

Corollary 1.2.4. *If* $u \in W^{1,p}(\Omega)$, *let* $\frac{1}{|\Omega|}\int_\Omega u dx = u_\Omega$. *Then there is a positive constant* C *depending only on* Ω, n *and* p *such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C(n,p) \|\nabla u\|_{L^p(\Omega)}. \tag{1.2.6}$$

Corollary 1.2.5. *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *Then there is a positive constant* C *depending only on* Ω, n *such that for any* $u \in H^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq C(n,p) \left(\|\nabla u\|_{L^2(\Omega)} + \left| \int_\Omega u dx \right| \right). \tag{1.2.7}$$

1.2.3 The Bellman–Gronwall Inequality

Some continuous integral inequalities of Gronwall type can be used in the analysis of various problems in the theory of nonlinear differential equations and integral equations. In 1919, Gronwall first proved the following famous inequality, which is called the Gronwall inequality.

Theorem 1.2.3. *Let* $u(t)$ *be a continuous function defined on the interval* $[t_0, t_1]$ *and*

$$u(t) \leq a + b \int_{t_0}^{t} u(s) ds, \tag{1.2.8}$$

where a *and* b *are non-negative constants. Then we have, for all* $t \in [t_0, t_1]$,

$$u(t) \leq a e^{b(t-t_0)}. \tag{1.2.9}$$

In 1943, Bellman extended the original Gronwall inequality, which reads in the following theorem.

Theorem 1.2.4. *Let a be a positive constant, $u(t)$ and $b(t) \geq 0$ be real-valued continuous functions on $t \in [t_0, t_1]$, satisfying*

$$u(t) \leq a + \int_{t_0}^{t} b(s)u(s)\,ds, \quad \forall t \in [t_0, t_1]. \tag{1.2.10}$$

Then we have

$$u(t) \leq a \exp\left\{\int_{t_0}^{t} b(s)\,ds\right\}, \quad \forall t \in [t_0, t_1]. \tag{1.2.11}$$

Generally, we use the following generalized Gronwall inequality.

Theorem 1.2.5. *Let $u(t)$, $a(t)$ and $b(t)$ be real-valued non-negative continuous functions defined for $[t_0, T)$, $T \leq +\infty$, $b(t) \geq 0$, satisfying*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(s)u(s)\,ds, \quad \forall t \in [t_0, T). \tag{1.2.12}$$

Then we have

$$u(t) \leq a(t) + \int_{t_0}^{t} a(s)b(s) \exp\left\{\int_{s}^{t} b(x)\,dx\right\} ds, \quad \forall t \in [t_0, T). \tag{1.2.13}$$

If, in addition, $a(t)$ is non-decreasing, then

$$u(t) \leq a(t) \exp\left\{\int_{t_0}^{t} b(s)\,ds\right\}, \quad \forall t \in [t_0, T). \tag{1.2.14}$$

1.2.4 The Hölder Inequality

The Hölder inequality plays an important role in the theory and applications of partial differential equations.

Theorem 1.2.6. *Suppose that $1 \leq p \leq +\infty$ and q denote the conjugate exponent defined by $q := \frac{p}{p-1}$, or equivalently $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are measurable functions, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{1.2.15}$$

Assuming $p \in (1, +\infty)$ and $\|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)} < +\infty$, equality holds in (1.2.15) if and only if $|f|^p$ and $|g|^q$ are linearly dependent as elements of $L^1(\Omega)$. If we further assume that $\|f\|_{L^p(\Omega)}$ and $\|g\|_{L^q(\Omega)}$ are positive, then equality holds in (1.2.15) if and only if

$$|g|^q\|f\|^p_{L^p(\Omega)} = \|g\|^q_{L^q(\Omega)}|f|^p, \quad a.e.\ in\ \Omega. \tag{1.2.16}$$

If $0 < p < 1$, *then*

$$\|fg\|_{L^1(\Omega)} \geq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}. \tag{1.2.17}$$

Corollary 1.2.6. *If* $p > 0$, $q > 0$ *and* $r > 0$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

and if $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$, *then* $fg \in L^r(\Omega)$ *and*

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}. \tag{1.2.18}$$

1.2.5 The Minkowski Inequality

In this subsection, we shall recall the Minkowski inequality.

Theorem 1.2.7. *If* $1 \leq p < +\infty$, f *and* g *are measurable functions,* $f \in L^p(\Omega)$ *and* $g \in L^p(\Omega)$, *then* $f + g \in L^p(\Omega)$ *and*

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{1.2.19}$$

The next theorem is about the Minkowski inequality for integrals.

Theorem 1.2.8. *Let* $1 \leq p < +\infty$. *Suppose that* f *is measurable on* $\mathbb{R}^m \times \mathbb{R}^n$, *that* $f(\cdot, y) \in L^p(\mathbb{R}^m)$ *for almost all* $y \in \mathbb{R}^m$, *and the function* $y \to \|f(\cdot, y)\|_{L^p(\mathbb{R}^m)}$ *belongs to* $L^1(\mathbb{R}^n)$. *Then the function* $x \to \int_{\mathbb{R}^n} f(x, y)\,dy$ *belongs to* $L^p(\mathbb{R}^m)$ *and*

$$\left(\int_{\mathbb{R}^m} |f(x, y)\,dy|^p\,dx\right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} |f(x, y)|^p\,dx\right)^{\frac{1}{p}} dy, \tag{1.2.20}$$

that is,

$$\left\|\int_{\mathbb{R}^m} f(\cdot, y)\,dy\right\|_{L^p(\mathbb{R}^m)} \leq \int_{\mathbb{R}^n} \|f(\cdot, y)\|_{L^p(\mathbb{R}^m)}\,dy.$$

1.2.6 The Jensen Inequality

Among all inequalities for convex functions, the Jensen inequality should be the most famous one.

Theorem 1.2.9. *Assume that the volume of* ω, $|\Omega| < +\infty$. *Let* $\Phi : \mathbb{R} \to (-\infty, +\infty]$ *be a convex lower semicontinuous function,* $\Phi \not\equiv +\infty$. *Let* $f \in L^1(\Omega)$ *be such that* $f(x) \in D(\Phi)$ *a.e. and* $\Phi(f) \in L^1(\Omega)$. *Then*

$$\Phi\left(\frac{1}{|\Omega|}\int_\Omega f(x)\,dx\right) \le \frac{1}{|\Omega|}\int_\Omega \Phi(f(x))\,dx. \tag{1.2.21}$$

Corollary 1.2.7. *Let* $\Phi(u) : [\alpha, \beta] \to \mathbb{R}$ *be a convex function. Suppose that* $a_k \geq 0$ $(k = 1, 2,\ldots, n)$ *are non-negative constants verifying* $\sum_{i=1}^n a_k > 0$, *then for any* x_1, $x_2,\ldots, x_n \in [\alpha, \beta]$, *we have*

$$\Phi\left(\frac{\sum_{i=1}^n a_k x_k}{\sum_{i=1}^n a_k}\right) \le \frac{\sum_{i=1}^n a_k \Phi(x_k)}{\sum_{i=1}^n a_k}.$$

Corollary 1.2.8. *Let* $\Phi(u) : [\alpha, \beta] \to \mathbb{R}$ *be a convex function. Suppose that* f: $t \in [a, b] \longrightarrow [\alpha, \beta]$, *and* $p(t)$ *are continuous functions with* $p(t) \geq 0$, $p(t) \not\equiv 0$. *Then*

$$\Phi\left(\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right) \le \frac{\int_a^b \Phi(f(t))p(t)\,dt}{\int_a^b p(t)\,dt}.$$

1.2.7 The Shen–Zheng Inequality

The next inequality is very useful and powerful in dealing with the global well-posedness and asymptotic behavior of solutions to some evolutionary differential equations.

Theorem 1.2.10. *([124]) Suppose that* $y(t)$, $h(t)$ *are non-negative functions,* $y'(t)$ *is locally integrable on* $(0, +\infty)$ *and* $y(t)$, $h(t)$ *satisfy the following conditions:*

$$\frac{dy(t)}{dt} \le A_1 y^2(t) + A_2 + h(t), \quad \forall t \geq 0, \tag{1.2.22}$$

$$\int_0^T y(t)\,dt \le A_3, \quad \int_0^T h(t)\,dt \le A_4, \quad \forall T > 0 \tag{1.2.23}$$

with $A_i (i = 1, 2, 3, 4)$ *being given non-negative constants independent of* t, T. *Then for any* $r > 0$, *the following estimates holds:*

$$y(t+r) \le \left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_3}, \quad \forall t \geq 0. \tag{1.2.24}$$

Furthermore, we have

$$\lim_{t \to +\infty} y(t) = 0. \tag{1.2.25}$$

Proof. Assume that $0 \leq t \leq t+r$ with any given $r>0$. Multiplying (1.2.22) by $\exp(-\int_t^s A_1 y(\tau)\,d\tau)$, we have

$$\frac{d}{ds}\left(y(s)e^{-\int_t^s A_1 y(\tau)\,d\tau}\right) \leq (A_2+h(s))e^{-\int_t^s A_1 y(\tau)\,d\tau} \leq A_2+h(s),$$

which, by integration between s and $t+r$, implies

$$y(t+r) \leq y(s)e^{\int_s^{t+r} A_1 y(\tau)\,d\tau} + (A_2 r + A_4)e^{\int_s^{t+r} A_1 y(\tau)\,d\tau} \leq (y(s)+A_2 r+A_4)e^{A_1 A_3}. \quad (1.2.26)$$

Integrating (1.2.26) with respect to s between t and $t+r$, we obtain (1.2.24). It follows from (1.2.22) and (1.2.24) that

$$\frac{dy}{dt} \leq A_1\left(\left(\frac{A_3}{r}+A_2 r+A_4\right)e^{A_1 A_3}\right)^2 + A_2 + h(t) = A_r + h(t), \quad \forall\, t \geq r, \quad (1.2.27)$$

where

$$A_r =: A_1\left(\left(\frac{A_3}{r}+A_2 r+A_4\right)e^{A_1 A_3}\right)^2 + A_2.$$

Now we prove (1.2.25) by the contradiction argument. Suppose that it is not true. Then there exists a monotone increasing sequence $\{t_n\}$ and a constant $a>0$ such that

$$\begin{aligned} & t_n \geq r + \frac{a}{4A_r}, \quad t_{n+1} \geq t_n + \frac{a}{4A_r}, \quad \forall\, n \in \mathbb{N}, \\ & \lim_{n\to+\infty} t_n = +\infty, \\ & y(t_n) \geq \frac{a}{2} > 0, \qquad \forall\, n \in \mathbb{N}. \end{aligned} \quad (1.2.28)$$

On the other hand, from (1.2.27) we obtain

$$y(t_n) - y(t) \leq A_r(t_n - t) + \int_t^{t_n} h(\tau)\,d\tau, \qquad \text{as } t_n - \frac{a}{4A_r} \leq t < t_n. \quad (1.2.29)$$

Combining (1.2.28) and (1.2.29) yields

$$\frac{a}{2} - y(t) \leq y(t_n) - y(t) \leq \frac{a}{4} + \int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau, \qquad \text{as } t_n - \frac{a}{4A_r} \leq t < t_n.$$

Therefore,

$$y(t) + \int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau \geq \frac{a}{4}, \qquad \text{as } t_n - \frac{a}{4A_r} \leq t < t_n. \quad (1.2.30)$$

Let

$$n_T = \max\left\{ n \in \mathbb{N} | r + \frac{a}{4A_r} \leq t \leq T \right\}.$$

Thus,

$$\lim_{T \to +\infty} n_T = +\infty.$$

It turns out from (1.2.30) that for all $T > 0$,

$$\begin{aligned} A_3 + \frac{aA_4}{4A_r} &\geq \int_0^T y(\tau)\,d\tau + \frac{a}{4A_r}\int_0^T h(\tau)\,d\tau \\ &\geq \sum_{1 \leq n \leq n_T} \left(\int_{t_n - \frac{a}{4A_r}}^{t_n} y(\tau)\,d\tau + \frac{a}{4A_r}\int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau \right) \\ &\geq \frac{a^2}{16A_r} n_T, \end{aligned}$$

which contradicts (1.2.23). □

In order to emphasise the importance of this inequality, we need to give the following uniform Gronwall inequality.

Theorem 1.2.11. *([43, 131]) Let g, h, y be three positive locally integrable functions on $(t_0, +\infty)$ such that y' is locally integrable on $(t_0, +\infty)$, and which satisfy, for $t \geq t_0$,*

$$\frac{dy(t)}{dt} \leq g(t)y(t) + h(t),$$

$$\int_t^{t+r} g(s)\,ds \leq a_1, \quad \int_t^{t+r} h(s)\,ds \leq a_2, \quad \int_t^{t+r} y(s)\,ds \leq a_3,$$

where $a_i (i = 1, 2, 3)$ and r are positive constants. Then for any $t \geq t_0$,

$$y(t + r) \leq \left(\frac{a_3}{r} + a_2 \right) e^{a_1}.$$

Compared with the above uniform Gronwall inequality, we easily see that

(1) The Shen–Zheng inequality generalizes the uniform Gronwall inequality.
(2) The Shen–Zheng inequality gives the asymptotic result which plays an important role in asymptotic analysis of solutions for some nonlinear evolutionary equations.

1.2.8 The Interpolation Inequality

The next interpolation inequality plays a very important role in the theory and applications of Lebesgue space.

Theorem 1.2.12. *Suppose that* $0 < p < q < r \leq +\infty$, *then* $L^p(\Omega) \cap L^r(\Omega) \subset L^q(\Omega)$ *and*

$$\|f\|_{L^q(\Omega)} \leq \|f\|^{\lambda}_{L^p(\Omega)} \|f\|^{1-\lambda}_{L^r(\Omega)}, \tag{1.2.31}$$

where $\lambda \in (0, 1)$ *is determined so that*

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

Furthermore, assume $1 \leq p < q < r \leq +\infty$, *and for* $f \in L^p(\Omega) \cap L^r(\Omega)$, *let*

$$\|f\| := \|f\|_{L^p(\Omega)} + \|f\|_{L^r(\Omega)}.$$

Then $(L^p(\Omega) \cap L^r(\Omega), \|\cdot\|)$ *is a Banach space and the inclusion map of* $L^p(\Omega) \cap L^r(\Omega)$ *into* L^q *is bounded, in fact,*

$$\|f\|_{L^q(\Omega)} \leq \max\Big(\lambda^{-1}, (1-\lambda)^{-1}\Big)\Big(\|f\|_{L^p(\Omega)} + \|f\|_{L^r(\Omega)}\Big), \tag{1.2.32}$$

where

$$\lambda = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}} = \frac{p(r-q)}{q(r-p)}.$$

1.2.9 The Gagliardo–Nirenberg Interpolation Inequality

The Gagliardo–Nirenberg interpolation inequality is a result in the theory of Sobolev spaces that estimate the weak derivatives of a function. The result is of particular importance in the theory of partial differential equations.

Theorem 1.2.13. *Let* u *belong to* $L^q(\mathbb{R}^n)$ *and its derivatives of order* m, $D^m u$, *belong to* $L^r(\mathbb{R}^n)$, $1 \leq q,\ r \leq +\infty$. *For the derivatives* $D^j u$, $0 \leq j < m$, *the following inequalities hold*

$$\left\|D^j u\right\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^n)}, \tag{1.2.33}$$

where

$$\frac{1}{p} - \frac{j}{n} = \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q},$$

for all α *in the interval*

$$\frac{j}{m} \leq \alpha \leq 1$$

and the constant $C > 0$ depending only on n, m, j, q, r, α, with the following exceptional cases:

(1) If $j = 0$, $rm < n$, $q = +\infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L^{\tilde{q}}(\mathbb{R}^n)$ for some finite $\tilde{q} > 0$.
(2) If $1 < r < +\infty$, and $m - j - \frac{n}{r}$ is a nonnegative integer, then (1.2.33) holds only for a satisfying $\frac{j}{m} \leq \alpha < 1$.

Remark 1.2.1. *For $\alpha = 1$, the fact that u is contained in $L^q(\mathbb{R}^n)$ does not enter in the estimate (1.2.33), and the estimate is equivalent to the results of Sobolev (note that we permit r to be unity).*

Remark 1.2.2. *For a bounded domain (with smooth boundary), the result holds if we add the term $C\|u\|_{L^{\tilde{q}}(\mathbb{R}^n)}$ to the right-hand side of (1.2.33) for any $\tilde{q} > 0$. The constants $C > 0$ then depend also on the domain.*

Remark 1.2.3. *Similar estimates hold for the L^p norms of $D^j u$ on linear subspaces of lower dimension, for suitable p.*

1.3 C_0-Semigroup

In this section, we shall recall some basic notations on C_0-semigroups of linear or nonlinear operators.

Definition 1.3.1. *Suppose that $(X, \|\cdot\|)$ is a Banach space and let $\mathbb{R}^+ = [0, +\infty)$. A mapping $S : \mathbb{R}^+ \times X \to X$, $(t, x) \mapsto S(t)x$, is said to be a C_0-semigroup (or a continuous semiflow, or a C_0-dynamical system) if*

(i) $S(0) = I$,
(ii) $S(s + t) = S(s)S(t)$, $s, t \in \mathbb{R}^+$,
(iii) The map $(t, x) \mapsto S(t)x$ is continuous in t, x for all $(t, x) \in \mathbb{R}^+ \times X$.

Definition 1.3.2. *A semigroup $S(t)$, $t \in \mathbb{R}^+$, is called a semigroup of contraction (or a non-expansive semigroup) if there exists a constant $\alpha \in (0, 1]$ such that*

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\| \quad \textit{for all } x, y \in X.$$

Definition 1.3.3. *If A is an operator with domain $D(A)$ consisting of points x such that the limit*

$$Ax = \lim_{h \to 0} \frac{S(h)x - x}{h}, \quad x \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup $S(t)$.

Remark 1.3.1. *Given an operator A, if A coincides with the infinitesimal generator of $S(t)$, then we say that it generates a C_0-semigroup $S(t),\ t \in \mathbb{R}^+$. Sometimes we also denote $S(t)$ by e^{At}.*

Definition 1.3.4. *e^{At} is said to be exponentially stable if there exist positive constants α and $M \geq 1$ such that*

$$\left\| e^{At} \right\| \leq M e^{-\alpha t}, \quad \forall\ t \in \mathbb{R}^+. \tag{1.3.1}$$

Notation: The notations $L^p(J)$ $(1 \leq p \leq +\infty)$ and $W^{k,p}(J)$ $(k \in \mathbb{N})$ denote the usual Lebesgue spaces and Sobolev spaces on the interval J. In particular, $W^{k,2}(J)$ is also denoted by $H^k(J)$, $H_0^1(J) = W_0^{1,2}(J)$. $\|\cdot\|_B$ denotes the norm in the space B, $\|\cdot\| := \|\cdot\|_{L^2}$. Analogously, spaces $L^p(J,\ B)$ represent the corresponding Lebesgue spaces of B-valued functions on the interval J. In addition, $C^\beta([0,\ T],\ B)$ denotes the Banach space of B-valued functions on $[0,\ T]$ which are uniformly Hölder continuous with exponent $\beta \in (0, 1)$. $C^{k+\beta}(\Omega)$, $k \in \mathbb{N}_0$ denotes the Banach space of functions on Ω which are k times uniformly Hölder continuous with exponent $\beta \in (0,\ 1)$ in variable $x \in \Omega$. ∂_t or $\frac{d}{dt}$ or a subscript t and ∂_x or a subscript x denote the derivative with respect to t and x in the distribution sense, respectively. The mean value $\bar{f}(t) = L^{-1} \int_0^L f(x, t)\,dx$. Letter C will denote the generic positive constant, which may vary from line to line, depending on the initial data and the physical coefficients of system but being independent of t.

Chapter 2

Global Existence and Exponential Stability of Spherically Symmetric Solutions to a Compressible Combustion Radiative and Reactive Gas

2.1 Introduction

Since combustion problems requiring theoretical analysis are primarily concerned with the flow of reacting and diffusing gases, we shall discuss the combustion fluid model of compressible radiative and reactive gases in this chapter. Nowadays, this is a hot topic in the field of applied mathematics, including the applied partial differential equations and computational mathematics, and fluid mechanics, laws of combustion phenomena, many researchers are engaged in this subject. So far, this is still a burgeoning research field.

Extremely rapid exothermic chemical reactions can develop in combustible materials. The interaction of the chemistry with the basic fluid flow is described by a highly nonlinear, extremely complex, degenerate, quasilinear parabolic system of partial differential equations. Some of this complexity is essential: the heat released by exothermic reaction is what makes a flame hot. David Leonard Chapman and Jacques Charles Emile Jouguet gave a simplified but convincing model which laid the foundation for combustion theory. This simplified assumption is that the reaction region is infinitely thin or, equivalently, the reaction rate is infinitely large, which means the chemical reaction occurs and completes instantaneously. Thus noting the difference in the chemical properties between the burned and the unburned gases, the expression of the total energy needs to be amended appropriately.

Generally speaking, the fluid micelle is usually a mixture of burned and unburned gases in the combustion process. However, the Chapman–Jouguet assumption is extremely idealized. Then the simplest way is to consider the mixed gases as a whole, a dilute mixture, whose fundamental physical properties (viscosity, thermal conductivity, etc.) are defined by a single component. Of course, this is a

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just method and still far from being accomplished. In this chapter, we shall focus on the corresponding system of partial differential equations for the radiative and reacting combustion fluid with viscosity and heat conduction.

2.2 Model and Main Results

2.2.1 Combustion Model of Radiative and Reactive Gas

In this subsection, we shall establish equations for this kind of mixed fluid as a whole and introduce the corresponding density ρ, velocity $\mathbf{U}$, pressure P, internal energy e, etc. Meanwhile, we also introduce a function $Z(x,\ t)$ to denote the reactant mass fraction of unburned gas (note that the completely unburned gas corresponds to $Z = 1$ and a totally burned gas corresponds to $Z = 0$).

Now we establish the reacting fluid dynamic model, which consists of Navier–Stokes equations corresponding to the conservation laws of the mass, the momentum and the energy coupling with the chemical kinetics equation in $\mathbf{\Omega} \times (0,\ +\infty)$ with a smooth bounded $\mathbf{\Omega} \subset \mathbb{R}^n$ (see, *e.g.*, [7, 35, 36, 80, 112, 134]). The equation of mass conservation is

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{U}) = 0. \tag{2.2.1}$$

The equation of momentum conservation is

$$\partial_t(\rho \mathbf{U}) + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P = \mathrm{div} S, \tag{2.2.2}$$

where the viscous stress tensor S characterizes the measure of resistance of the fluid.

The equation of energy conservation is

$$\partial_t(\rho e) + \mathrm{div}(\rho e \mathbf{U}) + \mathrm{div}\, Q = S : \nabla \mathbf{U} - P \mathrm{div} \mathbf{U} + k\lambda\phi\rho Z, \tag{2.2.3}$$

where the dissipative function $S : \nabla \mathbf{U}$ stands for a real dissipation of the mechanical energy into heat, $Q = Q(\rho,\ \theta)$ represents the heat flux and $\phi = \phi(\theta)$ means the reaction function. Positive constants k and λ are the coefficients of rate of the reactant and the difference in heat between the reactant and the product.

The chemical kinetics equation is

$$\partial_t(\rho Z) + \mathrm{div}(\rho \mathbf{U} Z) - \mathrm{div} F = -k\phi\rho Z, \tag{2.2.4}$$

where $\rho Z = \rho Z(x,\ t)$ represents the density of the reactant and F stands for the species diffusion velocity.

Now by the general principle on Newtonian fluids, the admissible form of S reads as

$$S = \mu(\nabla \mathbf{U} + \nabla \mathbf{U}^T) + \lambda_1 \mathrm{div} \mathbf{U} I, \tag{2.2.5}$$

where μ and λ_1 are called the viscosity coefficients satisfying

$$\mu > 0, \quad n\lambda_1 + 2\mu \geq 0. \tag{2.2.6}$$

Accordingly, $S : \nabla \mathbf{U}$ can be written as

$$S : \nabla \mathbf{U} = \sum_{i,j} \frac{\mu}{2}\left(\frac{\partial \mathbf{U}_i}{\partial x_j} + \frac{\partial \mathbf{U}_j}{\partial x_i}\right)^2 + \lambda_1 |\mathrm{div}\mathbf{U}|^2.$$

The reaction function $\phi = \phi(\theta)$ is defined by the first-order Arrhenius law (see, *e.g.*, [36])

$$\phi(\theta) = \theta^\beta e^{-\frac{A}{\theta}}, \tag{2.2.7}$$

where the positive constant A is the activation energy and β is a non-negative number. The species diffusion velocity F is assumed to satisfy Fickian law

$$F = D\nabla Z, \tag{2.2.8}$$

where $D = d\rho$ is the reactant flux diffusion coefficient and the positive constant d stands for the species diffusion in the reaction. The thermo-radiative flux Q satisfies the Fourier law

$$Q = Q(\rho, \theta) = -\kappa(\rho, \theta)\nabla\theta, \quad \kappa(\rho, \theta) = \kappa_1 + \kappa_2 \frac{\theta^q}{\rho}, \tag{2.2.9}$$

where κ_1, κ_2 and q are positive constants. Pressure P and internal energy e consist of a linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part

$$P = P(\rho, \theta) = R\rho\theta + \frac{a}{3}\theta^4, \quad e = e(\rho, \theta) = c_v\theta + a\frac{\theta^4}{\rho}, \tag{2.2.10}$$

where positive constants R, c_v and a are the perfect gas constant, the specific heat and the Stefan-Boltzmann constant, respectively (see, *e.g.*, [27]).

Now we consider the following boundary conditions

$$\mathbf{U}|_{\partial\Omega} = 0, \quad Q|_{\partial\Omega} = 0, \quad \nabla Z|_{\partial\Omega} = 0, \tag{2.2.11}$$

together with the initial conditions

$$(\rho, \mathbf{U}, \theta, Z)(x, 0) = (\rho_0, \mathbf{U}_0, \theta_0, Z_0)(x). \tag{2.2.12}$$

2.2.2 Spherical Symmetry Model of Combustion Model

Now we study the spherically symmetric solutions of the problem (2.2.1)–(2.2.12). First, we construct the corresponding system for radial solutions in the Eulerian coordinates. Let $r = |x|$ and take

$$\rho(x,t)=\rho(r,t),\quad \rho\mathbf{U}(x,t)=\rho u(r,t)\frac{x}{r},\quad \theta(x,t)=\theta(r,t),\quad Z(x,t)=Z(r,t). \tag{2.2.13}$$

Then we can obtain from the system (2.2.1)–(2.2.4) by the direct calculations

$$\begin{cases}
\rho_t+(\rho u)_r+\dfrac{n-1}{r}\rho u=0, & (2.2.14)\\
(\rho u)_t+(\rho u^2)_r+\dfrac{n-1}{r}\rho u^2+P_r=(2\mu+\lambda_1)\left(u_r+\dfrac{n-1}{r}u\right)_r, & (2.2.15)\\
(\rho e)_t+(\rho eu)_r+\dfrac{n-1}{r}\rho ue-\kappa_r\theta_r-\kappa\left(\theta_{rr}+\dfrac{n-1}{r}\theta_r\right) & \\
\quad =2\mu\left(u_r^2+\dfrac{n-1}{r^2}u^2\right)+\lambda_1\left(u_r+\dfrac{n-1}{r}u\right)^2-P\left(u_r+\dfrac{n-1}{r}u\right)+k\lambda\phi\rho Z, & (2.2.16)\\
(\rho Z)_t+(\rho uZ)_r+\dfrac{n-1}{r}\rho uZ=-k\phi\rho Z+DZ_{rr}+\dfrac{n-1}{r}DZ_r+D_rZ_r, & (2.2.17)
\end{cases}$$

where the domain $\mathbf{\Omega}$ is given by $\mathbf{\Omega}=\{x\in\mathbb{R}^n : a<|x|<b\}$ for some constants a and b with $0<a<b<+\infty$ and the initial boundary conditions (2.2.11) and (2.2.12) read as

$$u|_{r=a,b}=0,\quad \theta_r|_{r=a,b}=0,\quad Z_r|_{r=a,b}=0,\quad t\geq 0, \tag{2.2.18}$$

$$\rho(r,0)=\rho_0(r),\quad u(r,0)=u_0(r),\quad \theta(r,0)=\theta_0(r),\quad Z(r,0)=Z_0(r),\quad r\in(a,b). \tag{2.2.19}$$

Second, for our convenience, we may transform the system (2.2.14)–(2.2.19) into Lagrangian coordinates. The Eulerian coordinates $(r,\ t)$ are connected to the Lagrangian coordinates $(\xi,\ t)$ by the relation

$$r(\xi,t)=r_0(\xi)+\int_0^t\tilde{u}(\xi,\tau)\,d\tau, \tag{2.2.20}$$

where $\tilde{u}(\xi,\ t)=u(r(\xi,\ t),\ t)$ and

$$r_0(\xi)=\eta^{-1}(\xi),\quad \eta(r)=\int_a^r s^{n-1}\rho_0(s)\,ds,\quad r\in(a,b). \tag{2.2.21}$$

It then follows from equation (2.2.14) and boundary condition (2.2.18) that

$$\frac{\partial}{\partial t}\int_a^{r(\xi,t)} s^{n-1}\rho(s,t)\,ds=0. \tag{2.2.22}$$

Thus,

$$\int_a^{r(\xi,t)} s^{n-1}\rho(s,t)\,ds=\int_a^{r_0(\xi)} s^{n-1}\rho_0(s)\,ds=\xi, \tag{2.2.23}$$

and $\mathbf{\Omega}$ is transformed to $\mathbf{\Omega}_n = (0,\, L)$ with

$$L = \int_a^b s^{n-1}\rho_0(s)\,ds = \int_a^b s^{n-1}\rho(s,\, t)\,ds,$$

which, together with (2.2.20), (2.2.21) and (2.2.23), yields that L is invariant along the trajectory $\{\rho(s,\, t) : a \leq s \leq b,\ t \geq 0\}$. Moreover, we derive from (2.2.23) that

$$\partial_\xi r(\xi,\, t) = [r^{n-1}(\xi,\, t)\rho(r(\xi,\, t),\, t)]^{-1}. \tag{2.2.24}$$

In general, for any function $\varphi(r,\, t)$, if we denote $\tilde{\varphi}(\xi,\, t) := \varphi\ (r(\xi,\, t),\, t)$, then we derive from (2.2.20), (2.2.21) and (2.2.24) by the chain rule that

$$\partial_t\tilde{\varphi}(\xi,\, t) = \partial_t\varphi(r,\, t) + u\partial_r\varphi(r,\, t), \tag{2.2.25}$$

$$\partial_\xi\tilde{\varphi}(\xi,\, t) = \partial_r\varphi(r,\, t)\partial_\xi r(\xi,\, t) = \frac{\partial_r\varphi(r,\, t)}{r^{n-1}\rho(r,\, t)}. \tag{2.2.26}$$

Finally, without danger of confusion, we still denote $(\tilde{\rho},\, \tilde{u},\, \tilde{\theta})$ by $(\rho,\, u,\, \theta)$ and $(\xi,\, t)$ by $(x,\, t)$. We use $v := \frac{1}{\rho}$ to represent the specific volume. Thus by virtue of (2.2.20) and (2.2.24)–(2.2.26), equations (2.2.14)–(2.2.17) in the Eulerian coordinates can be rewritten in the Lagrangian coordinates in the new variables $(x,\, t)$, $x \in \mathbf{\Omega}_n$, $t \geq 0$ as follows

$$\begin{cases} v_t = (r^{n-1}u)_x, & (2.2.27)\\ u_t = r^{n-1}\left(\epsilon\dfrac{(r^{n-1}u)_x}{v} - P\right)_x, & (2.2.28)\\ e_t = \left(\dfrac{r^{2n-2}\kappa\theta_x}{v}\right)_x + \left(\epsilon\dfrac{(r^{n-1}u)_x}{v} - P\right)(r^{n-1}u)_x - 2\mu(n-1)(r^{n-2}u^2)_x + k\lambda\phi Z, & (2.2.29)\\ Z_t = d\left(\dfrac{r^{2n-2}Z_x}{v^2}\right)_x - k\phi Z, & (2.2.30) \end{cases}$$

with $\epsilon = \lambda_1 + 2\mu$ and the initial boundary conditions (2.2.18) and (2.2.19) become

$$u(x,t)|_{x=0,L} = 0, \quad \theta_x(x,t)|_{x=0,L} = 0, \quad Z_x(x,t)|_{x=0,L} = 0, \quad t \geq 0, \tag{2.2.31}$$

$$v(x,0) = v_0(x), \quad u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \quad Z(x,0) = Z_0(x), \quad x \in \mathbf{\Omega}_n. \tag{2.2.32}$$

Then by (2.2.20) and (2.2.24), $r(x,\, t)$ is determined by

$$r(x,t) = r_0(x) + \int_0^t u(x,s)\,ds, \quad r_0(x) := \left(a^n + n\int_0^x v_0(y)\,dy\right)^{\frac{1}{n}}, \ n = 2, 3,$$

that is,

$$r_t = u, \quad r^{n-1} r_x = v, \quad r|_{x=0} = a, \quad r|_{x=L} = b. \tag{2.2.33}$$

In this spherically symmetric model, we still consider pressure P, internal energy e and thermal conductivity κ are the linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part

$$P = R\frac{\theta}{v} + \frac{a}{3}\theta^4, \quad e = c_v\theta + av\theta^4, \quad \kappa = \kappa_1 + \kappa_2 v\theta^q. \tag{2.2.34}$$

2.2.3 Main Theorems

We define three classes of functions as

$$\begin{aligned}\mathcal{H}^1 = \{(v, u, \theta, Z) \in H^1[0, L] \times H^1[0, L] \times H^1[0, L] \times H^1[0, L] :\\ v(x) > 0, \theta(x) > 0, x \in [0, L], u(0) = u(L) = 0\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}^2 = \{(v, u, \theta, Z) \in H^2[0, L] \times H^2[0, L] \times H^2[0, L] \times H^2[0, L] : \ v(x) > 0, \theta(x) > 0,\\ x \in [0, L], u(0) = u(L) = 0, \theta'(0) = \theta'(L) = 0, Z'(0) = Z'(L) = 0\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^4 = \{(v, u, \theta, Z) \in H^4[0, L] \times H^4[0, L] \times H^4[0, L] \times H^4[0, L] : \ v(x) > 0, \theta(x) > 0,\\ x \in [0, L], u(0) = u(L) = 0, \theta'(0) = \theta'(L) = 0, Z'(0) = Z'(L) = 0\},\end{aligned}$$

which become the metric spaces when equipped with the metrics induced from the usual norms.

In what follows, letters C_i $(i = 1,\ 2,\ 4)$ will denote the universal constants depending on the norms of initial datum $(u_0,\ v_0,\ \theta_0,\ Z_0)$ in $\mathcal{H}^i$ but being independent of t, respectively.

Now we shall state our main theorems. Denote $E = \{(q, \beta) \in \mathbb{R}^2 : q \geq 2, 0 \leq \beta < q + 9\}$.

Theorem 2.2.1. *Let* $(q, \beta) \in E$. *Assume that there exists a constant* $\varepsilon_0 > 0$ *such that* $\overline{v_0} \equiv L^{-1}\int_0^L v_0(x)dx \leq \varepsilon_0$ *and the compatibility conditions are satisfied, then for the initial data* $(u_0, v_0, \theta_0, Z_0) \in \mathcal{H}^1$, *there exists a unique generalized global solution* $(u(x, t), v(x, t), \theta(x, t), Z(x, t)) \in \mathcal{H}^1$ *for all* $t \geq 0$ *to problem (2.2.27)–(2.2.32) verifying for all* $(x,\ t) \in [0,\ L] \times [0, +\infty)$,

$$0 < a \leq r(x, t) \leq b, \quad 0 < C_1^{-1} \leq v(x, t) \leq C_1, \quad 0 < C_1^{-1} \leq \theta(x, t) \leq C_1, \tag{2.2.35}$$

and

$$\|v(t)-\overline{v}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^1}^2+\|Z(t)\|_{H^1}^2+\|r(t)-\overline{r}\|_{H^2}^2+\|r_t(t)\|_{H^1}^2$$
$$+\int_0^t\Big(\|v-\overline{v}\|_{H^1}^2+\|u\|_{H^2}^2+\left\|\theta-\overline{\theta}\right\|_{H^2}^2+\|r-\overline{r}\|_{H^2}^2+\|r_t\|_{H^2}^2+\|Z\|_{H^2}^2$$
$$+\|v_t\|^2+\|u_t\|^2+\|\theta_t\|^2+\|Z_t\|^2\Big)(s)\,ds\leq C_1, \tag{2.2.36}$$

where $\overline{v}=\overline{v_0}=L^{-1}\int_0^L v_0(x)dx$*, the constant* $\overline{\theta}>0$ *is determined by* $e(\overline{v},\overline{\theta})=\int_0^L\left(\frac{1}{2}u_0^2+e(v_0,\theta_0)+\lambda Z_0\right)dx$ *and* $\overline{r}=(a^n+n\overline{v}x)^{\frac{1}{n}}$.

Moreover, there exists a constant $\gamma_1=\gamma_1(C_1)>0$ *such that for any fixed* $\gamma\in(0,\gamma_1]$ *and for any* $t>0$*, the following estimate holds*

$$e^{\gamma t}\Big(\|v(t)-\overline{v}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^1}^2+\|Z(t)\|_{H^1}^2+\|r(t)-\overline{r}\|_{H^2}^2+\|r_t(t)\|_{H^1}^2\Big)$$
$$+\int_0^t e^{\gamma s}\Big(\|v-\overline{v}\|_{H^1}^2+\|u\|_{H^2}^2+\left\|\theta-\overline{\theta}\right\|_{H^2}^2+\|r-\overline{r}\|_{H^2}^2+\|r_t\|_{H^2}^2+\|Z\|_{H^2}^2$$
$$+\|v_t\|^2+\|u_t\|^2+\|\theta_t\|^2+\|Z_t\|^2+\int_0^L\phi Z dx\Big)(s)\,ds\leq C_1. \tag{2.2.37}$$

Theorem 2.2.2. *Under the same assumptions as in theorem 2.2.1, and assume the compatibility conditions are satisfied, then for the initial data* $(u_0,v_0,\theta_0,Z_0)\in\mathcal{H}^2$*, there exists a unique generalized global solution* $(u(x,t),v(x,t),\theta(x,t),Z(x,t))\in\mathcal{H}^2$ *for all* $t\geq 0$ *to problem (2.2.27)–(2.2.32) verifying for all* $(x,\,t)\in[0,\,L]\times[0,\,+\infty)$,

$$\|v(t)-\overline{v}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^2}^2+\|Z(t)\|_{H^2}^2+\|r(t)-\overline{r}\|_{H^3}^2$$
$$+\|r_t(t)\|_{H^2}^2+\|u_t(t)\|^2+\|\theta_t(t)\|^2+\|Z_t(t)\|^2$$
$$+\int_0^t\Big(\|v-\overline{v}\|_{H^2}^2+\|u\|_{H^3}^2+\left\|\theta-\overline{\theta}\right\|_{H^3}^2+\|r-\overline{r}\|_{H^3}^2+\|r_t\|_{H^3}^2$$
$$+\|Z\|_{H^3}^2+\|u_{xt}\|^2+\|\theta_{xt}\|^2+\|Z_{xt}\|^2\Big)(s)\,ds\leq C_2, \tag{2.2.38}$$

where $\overline{v},\overline{\theta}>0$ *and* $\overline{r}$ *are the same as those in theorem 2.2.1.*

Moreover, there exists a constant $\gamma_2=\gamma_2(C_2)>0$ *such that for any fixed* $\gamma\in(0,\gamma_2]$ *and for any* $t>0$*, the following estimate holds*

$$e^{\gamma t}\Big(\|v(t)-\overline{v}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^2}^2+\|Z(t)\|_{H^2}^2$$
$$+\|r(t)-\overline{r}\|_{H^3}^2+\|r_t(t)\|_{H^2}^2+\|u_t(t)\|^2+\|\theta_t(t)\|^2+\|Z_t(t)\|^2\Big)$$
$$+\int_0^t e^{\gamma s}\Big(\|v-\overline{v}\|_{H^2}^2+\|u\|_{H^3}^2+\left\|\theta-\overline{\theta}\right\|_{H^3}^2+\|r-\overline{r}\|_{H^3}^2$$
$$+\|r_t\|_{H^3}^2+\|Z\|_{H^3}^2+\|u_{xt}\|^2+\|\theta_{xt}\|^2+\|Z_{xt}\|^2\Big)(s)\,ds\leq C_2. \tag{2.2.39}$$

Theorem 2.2.3. *Under the same assumptions as theorem 2.2.1, and assume the compatibility conditions are satisfied, then for the initial data* $(u_0, v_0, \theta_0, Z_0) \in \mathcal{H}^4$, *there exists a unique generalized global solution* $(u(x,t), v(x,t), \theta(x,t), Z(x,t)) \in \mathcal{H}^4$ *for all* $t \geq 0$ *to problem (2.2.27)–(2.2.32) verifying for all* $(x, t) \in [0, L] \times [0, +\infty)$,

$$\begin{aligned}&\|v(t)-\bar{v}\|_{H^4}^2+\|u(t)\|_{H^4}^2+\left\|\theta(t)-\bar{\theta}\right\|_{H^4}^2+\|Z(t)\|_{H^4}^2+\|r(t)-\bar{r}\|_{H^5}^2+\|r_t(t)\|_{H^4}^2+\|u_t(t)\|_{H^2}^2\\&\quad+\|\theta_t(t)\|_{H^2}^2+\|Z_t(t)\|_{H^2}^2+\int_0^t\Big(\|v-\bar{v}\|_{H^4}^2+\|u\|_{H^5}^2+\left\|\theta-\bar{\theta}\right\|_{H^5}^2+\|r-\bar{r}\|_{H^5}^2+\|r_t\|_{H^5}^2\\&\quad+\|Z\|_{H^5}^2+\|u_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2+\|Z_t\|_{H^3}^2+\|u_{tt}\|_{H^1}^2+\|\theta_{tt}\|_{H^1}^2+\|Z_{tt}\|_{H^1}^2\Big)(s)\,ds\leq C_4,\end{aligned} \tag{2.2.40}$$

where $\bar{v}, \bar{\theta} > 0$ *and* $\bar{r}$ *are the same as those in theorem 2.2.1.*

Moreover, there exists a constant $\gamma_4 = \gamma_4(C_4) > 0$ *such that for any fixed* $\gamma \in (0, \gamma_4]$ *and for any* $t > 0$, *the following estimate holds*

$$\begin{aligned}&e^{\gamma t}\Big(\|v(t)-\bar{v}\|_{H^4}^2+\|u(t)\|_{H^4}^2+\left\|\theta(t)-\bar{\theta}\right\|_{H^4}^2+\|Z(t)\|_{H^4}^2+\|r(t)-\bar{r}\|_{H^5}^2+\|r_t(t)\|_{H^4}^2+\|u_t(t)\|_{H^2}^2\\&\quad+\|\theta_t(t)\|_{H^2}^2+\|Z_t(t)\|_{H^2}^2\Big)+\int_0^t e^{\gamma s}\Big(\|v-\bar{v}\|_{H^4}^2+\|u\|_{H^5}^2+\left\|\theta-\bar{\theta}\right\|_{H^5}^2+\|r-\bar{r}\|_{H^5}^2+\|r_t\|_{H^5}^2\\&\quad+\|Z\|_{H^5}^2+\|u_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2+\|Z_t\|_{H^3}^2+\|u_{tt}\|_{H^1}^2+\|\theta_{tt}\|_{H^1}^2+\|Z_{tt}\|_{H^1}^2\Big)(s)\,ds\leq C_4.\end{aligned} \tag{2.2.41}$$

The results of this chapter are selected from [120]. Our present results include some novelties as follows.

(1) We establish the global existence, uniqueness and exponential stability of spherically symmetric solutions to our problem. To the best of our knowledge, this is the first result for this model.

(2) Our main results hold for $E = \{(q, \beta) \in \mathbb{R}^2 : q \geq 2,\ 0 \leq \beta < q + 9\}$, which improves the previous works [110, 112, 134, 135].

(3) In our model, we need the assumption of initial data $(v_0, u_0, \theta_0, Z_0) \in \mathcal{H}^i$ $(i = 1, 2, 4)$. Besides, we also need the smallness assumption of initial datum $v_0 : \overline{v_0} = L^{-1} \int_0^L v_0(x)\,dx \leq \varepsilon$.

2.3 Global Existence and Exponential Stability in $\mathcal{H}^1$

In this section, we shall establish the global existence and exponential stability in $\mathcal{H}^1$ for the generalized solutions to problem (2.2.27)–(2.2.32). That is to say, we shall complete the proof of theorem 2.2.1, which can be divided into the series of lemmas.

By the same arguments as those in [99, 109], we can derive from (2.2.33) that for all $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < a = r(0, t) \leq r(x, t) \leq r(L, t) = b. \tag{2.3.1}$$

2.3.1 Global Existence of Solutions in $\mathcal{H}^1$

In this subsection, we establish the global existence of solutions in $\mathcal{H}^1$. First, we establish some basic equalities or inequalities.

Lemma 2.3.1. *Under assumptions in theorem 2.2.1, there exists a constant $C_1 > 0$ such that the following estimates hold for any $t > 0$,*

$$L^{-1}\int_0^L v(x,t)\,dx = L^{-1}\int_0^L v_0(x)\,dx := \overline{v_0}, \tag{2.3.2}$$

$$\int_0^L \left(e + \frac{u^2}{2} + \lambda Z\right)(x,t)\,dx = \int_0^L \left(e_0 + \frac{u_0^2}{2} + \lambda Z_0\right)(x)\,dx := E_0, \tag{2.3.3}$$

$$\theta(x,t) > 0 \quad 0 \leq Z(x,t) \leq 1, \tag{2.3.4}$$

$$U(t) + \int_0^t V(s)\,ds \leq C_1, \tag{2.3.5}$$

$$\int_0^L Z(x,t)\,dx + k\int_0^t\int_0^L \phi(x,s)Z(x,s)\,dxds = \int_0^L Z_0(x)\,dx, \tag{2.3.6}$$

$$\int_0^L Z^2(x,t)\,dx + \int_0^t\int_0^L \left(\frac{Z_x^2}{v^2} + \phi Z^2\right)(x,s)\,dxds \leq C_1, \tag{2.3.7}$$

where

$$\begin{cases} U(t) = \displaystyle\int_0^L \left(\frac{u^2}{2} + R(v - \log v - 1) + C_v(\theta - \log\theta - 1)\right)(x,t)\,dx, & (2.3.8)\\ V(t) = \displaystyle\int_0^L \left(\frac{u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2} + \frac{k\lambda\phi Z}{\theta}\right)(x,t)\,dx. & (2.3.9)\end{cases}$$

Proof. We can easily get (2.3.2) and (2.3.6) by integrating (2.2.27) and (2.2.30), respectively, over $Q_t := (0, L) \times (0, t)$. Multiplying (2.2.28) by u, (2.2.30) by λ, respectively, and adding the results to (2.2.29) and then integrating the result over Q_t, we obtain (2.3.3).

Equation (2.2.29) can be written as

$$e_\theta\theta_t + \theta P_\theta(r^{n-1}u)_x = \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x + \epsilon\frac{(r^{n-1}u)_x^2}{v} - 2\mu(n-1)(r^{n-2}u^2)_x + k\lambda\phi Z. \tag{2.3.10}$$

We can derive from the maximum principle and the positivity of θ_0 that $\theta(x, t) > 0$.

Define the parabolic operator $\mathcal{L}$ by

$$\mathcal{L}u \equiv -\frac{\partial u}{\partial t} + d\frac{\partial}{\partial t}\left(\frac{r^{2n-2}}{v^2}\frac{\partial u}{\partial x}\right) - k\phi(\theta)u,$$

then by (2.2.30), $Z(x, t)$ satisfies

$$\begin{cases} \mathcal{L}Z \le 0, & \\ Z(x,0) \ge 0, & x \in [0, L], \\ \partial_x Z \ge 0, & x = 0, L. \end{cases}$$

Applying the maximal principle for the operator $\mathcal{L}$, we know that for all $(x, t) \in Q_t$,

$$Z(x, t) \ge 0.$$

Similarly, we take $\tilde{Z} \equiv 1 - Z$ and can show that for all $(x, t) \in Q_t$, $Z(x, t) \le 1$. Multiplying (2.3.10) by $\frac{1}{\theta}$ and integrating by parts, we can obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^L \left(c_v \log\theta + R\log v + \frac{4}{3}av\theta^3\right)dx \\ &\quad = \int_0^L \left(\frac{r^{2n-2}\kappa\theta_x^2}{v\theta^2} + \frac{k\lambda\phi Z}{\theta} + \epsilon\frac{(r^{n-2}u)_x^2}{v\theta} - 2\mu(n-1)\frac{(r^{n-2}u^2)_x}{\theta}\right)dx. \end{aligned} \tag{2.3.11}$$

It follows from (2.2.8) and (2.2.33) that

$$\begin{aligned} &\epsilon\frac{(r^{n-2}u)_x^2}{v\theta} - 2\mu(n-1)\frac{(r^{n-2}u^2)_x}{\theta} \\ &\quad = \frac{1}{v\theta}\left((n-1)(2\mu + (n-1)\lambda_1)\left(r^{-1}uv + \frac{\lambda_1 r^{n-1}u_x}{2\mu + (n-1)\lambda_1}\right)^2\right. \\ &\qquad \left. + \frac{2\mu(2\mu + n\lambda_1)}{2\mu + (n-1)\lambda_1}r^{2n-2}u_x^2\right) \ge \frac{2\mu(2\mu + n\lambda_1)}{2\mu + (n-1)\lambda_1}\frac{r^{2n-2}u_x^2}{v\theta}. \end{aligned} \tag{2.3.12}$$

Integrating (2.3.11) over Q_t and combining it with (2.3.3) and (2.3.12), we obtain

$$U(t) + \int_0^t V(s)ds \le C_1\left(1 + \int_0^L v\theta^3 dx\right). \tag{2.3.13}$$

By virtue of the Hölder inequality, we easily get, for any $0 \le m \le 4$,

$$\int_0^L v\theta^m dx \le \left(\int_0^L v\theta^4 dx\right)^{\frac{m}{4}}\left(\int_0^L vdx\right)^{\frac{4-m}{4}} \le C_1. \tag{2.3.14}$$

Thus estimate (2.3.5) holds. Relation (2.3.7) can be obtained by multiplying (2.2.30) by Z and integrating the result over Q_t. The proof is now complete. $\square$

Remark 2.3.1. *By the Jensen inequality, the mean value theorem and (2.3.5), we can know that there exists a point $b(t) \in [0, L]$ and two positive constants α_1, α_2 such that*

$$0 < \alpha_1 \leq \tilde{\theta}(t) := L^{-1} \int_0^L \theta(x, t)\, dx = \theta(b(t), t) \leq \alpha_2, \tag{2.3.15}$$

where α_1, α_2 are two roots of the equation $y - \log y - 1 = \frac{C_1}{c_v}$.

The next two lemmas concern the uniform upper or lower bound of the specific volume v. We first establish the expression of v in lemma 2.3.2 and then estimate the uniform upper or lower bound of v in lemma 2.3.3 based on this expression.

Lemma 2.3.2. *For any $t \geq 0$, there exists one point $x_1 = x_1(t) \in [0, L]$ such that the solution $v(x, t)$ to problem (2.2.27)–(2.2.32) can be expressed as for any $\delta \geq 0$,*

$$v(x, t) = \frac{D(x, t)}{B(x, t)} \left(1 + \epsilon^{-1} \int_0^t \frac{B(x, s) v(x, s)(P(x, s) - \delta)}{D(x, s)} ds\right), \tag{2.3.16}$$

where

$$\begin{aligned} B(x, t) = \exp \Bigg[\epsilon^{-1} \Bigg(& \frac{1}{\overline{v_0}} \int_0^t \int_0^L \left(\frac{u^2}{n} + Pv\right)(x, s)\, dx ds \\ & + \frac{(n-1)a^n}{n\overline{v_0}} \int_0^t \int_0^L (r^{-n} u^2)(x, s)\, dx ds \\ & + (n-1) \int_0^t \int_x^L (r^{-n} u^2)(y, s)\, dy ds - \delta t \Bigg) \Bigg], \end{aligned} \tag{2.3.17}$$

$$\begin{aligned} D(x, t) = v_0(x) \exp \Bigg[\epsilon^{-1} \Bigg(& \frac{1}{\overline{v_0}} \int_0^L v_0(x) \int_0^x (r_0^{1-n} u_0)(y)\, dy dx \\ & - \int_0^x (r_0^{1-n} u_0)(y)\, dy + \int_{x_1(t)}^x (r^{1-n} u)(y, t)\, dy \Bigg) \Bigg]. \end{aligned} \tag{2.3.18}$$

Proof. The main idea of the proof is similar to that of lemma 5.2.3 in Qin and Huang [109]. However, we note that our equation (2.2.28) is more complicated than that in Qin and Huang [109]. Moreover, here $B(x, t)$ in the expression of $v(x, t)$ does not only depend on t but also on x. Thus we need a detailed analysis.

First, let $\sigma(x, t) = \epsilon \frac{\left(r^{n-1} u\right)_x}{v} - P$, then for any $\delta \geq 0$, we can rewrite (2.2.27) as

$$v_t - \frac{1}{\epsilon}(\sigma + \delta) v = \frac{1}{\epsilon}(P - \delta) v. \tag{2.3.19}$$

Multiplying (2.3.19) by $\exp\Big(-\frac{1}{\epsilon}\int_0^t(\sigma+\delta)(x,s)\,ds\Big)$, we derive that

$$v(x,t)=\exp\left(\frac{1}{\epsilon}\int_0^t(\sigma+\delta)\,ds\right)\left(v_0+\frac{1}{\epsilon}\int_0^t v(P-\delta)\exp\left(-\frac{1}{\epsilon}\int_0^s(\sigma+\delta)\,d\tau\right)ds\right). \tag{2.3.20}$$

Second, we denote

$$h(x,t)=\int_0^t\sigma(x,s)\,ds+\int_0^x\big(r_0^{1-n}u_0\big)(y)\,dy+(n-1)\int_0^t\int_x^L\big(r^{-n}u^2\big)(y,s)\,dyds, \tag{2.3.21}$$

then

$$h_x=r^{1-n}u,\quad h_t=\sigma+(n-1)\int_x^L r^{-n}u^2\,dy=\epsilon\frac{(r^{n-1}u)_x}{v}-P+(n-1)\int_x^L r^{-n}u^2\,dy \tag{2.3.22}$$

which, along with (2.2.27) and (2.2.31), implies

$$(vh)_t=h\big(r^{n-1}u\big)_x+\epsilon\big(r^{n-1}u\big)_x-Pv+\frac{n-1}{n}(r^n)_x\int_x^L r^{-n}u^2\,dy, \tag{2.3.23}$$

$$h_x|_{x=0,L}=0. \tag{2.3.24}$$

Integrating (2.3.23) over Q_t and using (2.2.31) and (2.3.24), we can get

$$\int_0^L vh\,dx=\int_0^L v_0h_0\,dx-\int_0^t\int_0^L\left(\frac{u^2}{n}+Pv\right)dxds-\frac{n-1}{n}a^n\int_0^t\int_0^L r^{-n}u^2\,dxds$$

which, by the mean value theorem, implies that there exists a point $x_1(t)\in[0,L]$ for any $t\geq 0$ such that

$$h(x_1(t),t)=\frac{1}{\overline{v_0}}\left[\int_0^L v_0h_0\,dx-\int_0^t\int_0^L\left(\frac{u^2}{n}+Pv\right)dxds-\frac{n-1}{n}a^n\int_0^t\int_0^L r^{-n}u^2\,dxds\right]. \tag{2.3.25}$$

Thus by (2.3.21)–(2.3.25), we can derive

$$\begin{aligned}\int_0^t\sigma(x_1(t),s)\,ds=&\frac{1}{\overline{v_0}}\left[\int_0^L v_0h_0\,dx-\int_0^t\int_0^L\left(\frac{u^2}{n}+Pv\right)dxds\right.\\&\left.-\frac{n-1}{n}a^n\int_0^t\int_0^L r^{-n}u^2\,dxds\right]\\&-\int_0^{x_1(t)}r_0^{1-n}u_0\,dy-(n-1)\int_0^t\int_{x_1(t)}^L\big(r^{-n}u^2\big)(y,s)\,dyds.\end{aligned} \tag{2.3.26}$$

Finally, integrating (2.2.28) over $[x_1(t), x] \times [0, t]$, we derive for any fixed $t \geq 0$,

$$\int_0^t \sigma(x, s)\,ds = \int_0^t \sigma(x_1(t), s)\,ds + \int_{x_1(t)}^x \left(r^{1-n}u - r_0^{1-n}u_0\right)dy \\ + (n-1)\int_0^t \int_{x_1(t)}^x r^{-n}u^2\,dyds. \tag{2.3.27}$$

Then we derive from (2.3.22), (2.3.26) and (2.3.27) that

$$\begin{aligned}\int_0^t \sigma(x, s)\,ds &= \frac{1}{\overline{v_0}}\left[\int_0^L v_0 h_0\,dx - \int_0^t \int_0^L \left(\frac{u^2}{n} + Pv\right)dxds - \frac{n-1}{n}a^n \int_0^t \int_0^L r^{-n}u^2\,dxds\right] \\ &\quad - \int_0^{x_1(t)} r_0^{1-n}u_0\,dy + \int_{x_1(t)}^x (r^{1-n}u - r_0^{1-n}u_0)\,dy + (n-1)\int_0^t \int_{x_1(t)}^x r^{-n}u^2\,dyds \\ &= \frac{1}{\overline{v_0}}\left[\int_0^L v_0 h_0\,dx - \int_0^t \int_0^L \left(\frac{u^2}{n} + Pv\right)dxds - \frac{n-1}{n}a^n \int_0^t \int_0^L r^{-n}u^2\,dxds\right] \\ &\quad - \int_0^x r_0^{1-n}u_0\,dy + \int_{x_1(t)}^x r^{1-n}u\,dy - (n-1)\int_0^t \int_x^L r^{-n}u^2\,dyds\end{aligned}$$

which, together with (2.3.17) and (2.3.18), leads to

$$\frac{D(x, t)}{B(x, t)} = v_0(x)\exp\left(\frac{1}{\epsilon}\int_0^t (\sigma + \delta)\,ds\right) \tag{2.3.28}$$

which, along with (2.3.20), implies (2.3.16). The proof is complete. □

Lemma 2.3.3. *Under assumptions in theorem 2.2.1, the following estimate holds for all* $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \leq v(x, t) \leq C_1. \tag{2.3.29}$$

Proof. The main idea of the proof is similar to that of lemma 5.2.4 in Qin and Huang [109]. Noting the expressions of pressure P and internal energy e, we can derive from (2.2.34) and (2.3.3) that

$$\begin{aligned}\int_0^L vP\,dx &\leq \max\left\{R, \frac{a}{3}\right\}\int_0^L (\theta + v\theta^4)\,dx \\ &\leq \frac{\max\{R, \frac{a}{3}\}}{\min\{c_v, a\}}\int_0^L e\,dx \leq \frac{\max\{R, \frac{a}{3}\}}{\min\{c_v, a\}}E_0\end{aligned}$$

and

$$\int_0^L u^2\,dx \leq 2E_0$$

which, along with (2.3.15), implies

$$0 < a_1 \le \frac{1}{\epsilon \overline{v_0}} \int_0^L \left(vP + \frac{u^2}{n} \right)(x, t)\, dx \le a_2 \tag{2.3.30}$$

where

$$a_1 = \frac{R\alpha_1}{\epsilon \overline{v_0}}, \quad a_2 = \frac{1}{\epsilon \overline{v_0}} \left[R\alpha_2 + \left(\frac{\max\{R, \frac{a}{3}\}}{\min\{c_v, a\}} + \frac{2}{n} \right) E_0 \right].$$

Using lemmas 2.3.1 and 2.3.2, we derive for any $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \le D(x, t) \le C_1. \tag{2.3.31}$$

On the other hand, from lemma 2.3.1 for $0 < m_1 \le 2$, it follows that

$$\begin{aligned} |\theta^{m_1}(x, t) - \theta^{m_1}(b(t), t)| &\le C_1 \left| \int_{b(t)}^x \theta^{m_1 - 1} \theta_x\, dy \right| \\ &\le C_1 \left(\int_0^L \frac{\kappa(v, \theta)\theta_x^2}{v\theta^2} dx \right)^{\frac{1}{2}} \left(\int_0^L \frac{v\theta^{2m_1}}{\kappa(v, \theta)} dx \right)^{\frac{1}{2}} \\ &\le C_1 V_1^{\frac{1}{2}}(t) \left(\int_0^L \frac{v(1 + \theta^4)}{\kappa_1} dx \right)^{\frac{1}{2}} \\ &\le C_1 V_1^{\frac{1}{2}}(t) \end{aligned} \tag{2.3.32}$$

where $V_1(t) = \int_0^L \frac{\kappa(v, \theta)\theta_x^2}{v\theta^2} dx$. Thus for $0 < m_1 \le 2$,

$$\frac{1}{2}\alpha_1^{2m_1} - C_1 V_1(t) \le \theta^{2m_1}(x, t) \le 2\alpha_2^{2m_1} + C_1 V_1(t). \tag{2.3.33}$$

Obviously, we can derive from (2.3.30) and lemmas 2.3.1 and 2.3.2, for $\delta \ge 0$ and $0 \le s \le t$,

$$\exp\left(-\left(a_2 - \frac{\delta}{\epsilon} \right)(t - s) \right) \le \frac{B(x, t)}{B(x, s)} \le \exp\left(-\left(a_1 - \frac{\delta}{\epsilon} \right)(t - s) \right). \tag{2.3.34}$$

Thus for $\delta = 0$ in lemma 2.3.2, we can derive from (2.3.31)–(2.3.34) that there exists a large time $t_0 > 0$ such that as $t \ge t_0 > 0$,

$$\begin{aligned} v(x, t) &\ge C_1^{-1} \int_0^t \theta e^{-a_2(t-s)} ds \\ &\ge C_1^{-1} \int_0^t \left(\frac{1}{2}\alpha_1^{2m_1} - C_1 V_1(s) \right) e^{-a_2(t-s)} ds \\ &\ge \frac{\alpha_1^2}{2a_2 C_1}(1 - e^{-a_2 t}) - C_1 \int_0^t V_1(s) e^{-a_2(t-s)} ds \\ &\ge \frac{\alpha_1^2}{4a_2 C_1} > 0. \end{aligned} \tag{2.3.35}$$

In addition, we can also derive from lemma 2.3.2 and (2.3.34) (for $s = 0$) that for $\delta = 0$ and for any $t \in [0,\, t_0]$,

$$v(x,\,t) \geq \frac{D(x,\,t)}{B(x,\,t)} \geq C_1^{-1} e^{-a_2 t} \geq C_1^{-1} e^{-a_2 t_0} > 0$$

which, together with (2.3.35), gives us that for all $(x,\,t) \in [0,\, L] \times [0,\, +\infty)$,

$$v(x,\,t) \geq C_1^{-1} > 0. \tag{2.3.36}$$

Now we begin to show the positive lower bound of v. To prove it, we need the smallness of the total initial mass $\overline{v_0}$, which is only used to prove the positive lower bound of v.

We choose $0 < \varepsilon_0 \leq \frac{3R\alpha_1}{a\alpha_2^4}$, then as $\overline{v_0} \leq \varepsilon_0$, we have

$$\frac{R\alpha_1}{\overline{v_0}} \geq \frac{R\alpha_1}{\varepsilon_0} > \frac{a\alpha_2^4}{3} \geq \frac{a\tilde{\theta}^4}{3},$$

which implies that we can pick $\delta > 0$ such that

$$\frac{R\alpha_1}{\overline{v_0}} \geq \frac{R\alpha_1}{\varepsilon_0} > \delta > \frac{a\alpha_2^4}{3} \geq \frac{a\tilde{\theta}^4}{3}.$$

Thus we know

$$\xi_1 \equiv a_1 - \frac{\delta}{\epsilon} = \frac{1}{\epsilon}\left(\frac{R\alpha_1}{\overline{v_0}} - \delta\right) > 0, \tag{2.3.37}$$

$$\xi_2 \equiv \frac{3}{a}\tilde{\theta}^{-4}\delta - 1 \geq \frac{3}{a}\widetilde{\alpha_2}^{-4}\delta - 1 \equiv \xi_0 > 0. \tag{2.3.38}$$

Noting that by the Young inequality with ε and inequality $b^4 - 1 - \varepsilon \leq (b^2 - 1)^2 \times (1 + \varepsilon^{-1})$ with $b \in \mathbb{R}$ and any $\varepsilon > 0$, we derive from (2.3.37) and (2.3.38) that

$$v(P - \delta) = R\theta + v\Big(\frac{a}{3}\theta^4 - \delta\Big) \tag{2.3.39}$$

and

$$\begin{aligned} v\Big(\frac{a}{3}\theta^4 - \delta\Big) &= \frac{a}{3}\tilde{\theta}^4 v\Big(\hat{\theta}^4 - \frac{3}{a}\tilde{\theta}^{-4}\delta\Big) \\ &\leq \frac{a}{3}\tilde{\theta}^4 v \max\Big(\hat{\theta}^4 - 1 - \xi_2, 0\Big) \\ &\leq \frac{a}{3}\tilde{\theta}^4 v \max\Big(\hat{\theta}^4 - 1 - \xi_0, 0\Big) \\ &\leq \frac{a}{3}\tilde{\theta}^4 v\Big(\hat{\theta}^2 - 1\Big)^2(1 + \xi_0^{-1}) \\ &\leq C_1 v\Big(\hat{\theta}^2 - 1\Big)^2 \end{aligned} \tag{2.3.40}$$

where $\hat{\theta}(x,t)=\frac{\theta(x,t)}{\tilde{\theta}}$ and $\tilde{\theta}$ is defined by (2.3.15) in remark 2.3.1. Noting that $\hat{\theta}(b(t),t)=1$ and by the Poincaré inequality, we derive

$$\begin{aligned}(\hat{\theta}^2-1)^2 &\le \left(\int_0^L 2|\hat{\theta}\hat{\theta}_x|\,dx\right)^2 \le C_1\left(\int_0^L |\theta\theta_x|\,dx\right)^2\\ &\le C_1\left(\int_0^L \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx\right)\left(\int_0^L \frac{v\theta^4}{\kappa(v,\theta)}dx\right)\\ &\le C_1 V_1(t)\int_0^L v\theta^4\,dx \le C_1 V_1(t)\end{aligned}$$

which, together with (2.3.39) and (2.3.40), gives

$$v(P-\delta)\le R\theta + C_1 V_1(t)v. \tag{2.3.41}$$

Thus it follows from lemma 2.3.2, (2.3.31)–(2.3.33), (2.3.40) and (2.3.41) that

$$\begin{aligned}v(x,t) &\le C_1 + C_1\int_0^t (\theta(x,s)+v(x,s)V_1(s))\exp(-\xi_1(t-s))\,ds\\ &\le C_1 + C_1\int_0^t (1+V_1(s)+v(x,s)V_1(s))\exp(-\xi_1(t-s))\,ds\\ &\le C_1 + C_1\int_0^t v(x,s)V_1(s)\exp(-\xi_1(t-s))\,ds,\end{aligned}$$

that is,

$$F(t)\le C_1 e^{\xi_1 t} + C_1\int_0^t V_1(s)F(s)\,ds \tag{2.3.42}$$

with $F(t)=e^{\xi_1 t}\max_{x\in[0,L]} v(x,t)\equiv e^{\xi_1 t}M_v(t)$. Therefore, by the Bellman–Gronwall inequality, we have

$$F(t)\le C_1 e^{\xi_1 t}\exp\left(C_1\int_0^t V_1(s)\,ds\right)\le C_1 e^{\xi_1 t},$$

that is,

$$M_v(t)\le C_1$$

which, along with (2.3.36), completes the proof of the lemma. □

The next corollary is the key estimate in our proofs which can be obtained from lemmas 2.3.1 and 2.3.3.

Corollary 2.3.1. *Under assumptions in theorem 2.2.1, the following estimate holds for any* $(x,t)\in[0,L]\times[0,+\infty)$,

$$C_1^{-1}-C_1V_1(t)\le\theta^{2m}(x,t)\le C_1+C_1V_1(t) \tag{2.3.43}$$

with $V_1(t) = \int_0^L \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx$ *and* $0 \le m \le \frac{q+4}{2}$.

The next lemma is aimed at the estimate on v in H^1.

Lemma 2.3.4. *Under assumptions in theorem 2.2.1, the following estimate holds for any* $t > 0$,

$$\left\| r^{1-n}u - \epsilon\frac{v_x}{v}\right\|^2 + \int_0^t \int_0^L \left(v_x^2 + \theta v_x^2\right)(x,s)\,dxds \le C_1 + C_1 A^{\frac{q_0}{2}} \tag{2.3.44}$$

with $A := \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}$ *and* $q_0 = \max\{4-q, 0\}$.

Proof. Equation (2.2.28) can be rewritten as

$$\left(r^{1-n}u - \epsilon\frac{v_x}{v}\right)_t = -P_x - \frac{(n-1)u^2}{r^n}. \tag{2.3.45}$$

Multiplying (2.3.45) by $r^{1-n}u - \epsilon\frac{v_x}{v}$ and then integrating the resultant over Q_t, we have

$$\begin{aligned}
&\frac{1}{2}\left\| r^{1-n}u - \epsilon\frac{v_x}{v}\right\|^2 + R\epsilon\int_0^t\int_0^L \frac{\theta v_x^2}{v^3}dxds \\
&\quad= \frac{1}{2}\left\| r_0^{1-n}u_0 - \epsilon\frac{v_{0x}}{v_0}\right\|^2 + \int_0^t\int_0^L \left[R\frac{u\theta v_x}{r^{n-1}v^2} - \left(\frac{R}{v} + \frac{4}{3}a\theta^3\right)\theta_x\left(r^{1-n}u - \epsilon\frac{v_x}{v}\right)\right. \\
&\qquad\left. -(n-1)\frac{u^2}{r^n}\left(r^{1-n}u - \epsilon\frac{v_x}{v}\right)\right]dxds
\end{aligned}$$

whence

$$\begin{aligned}
&\left\| r^{1-n}u - \epsilon\frac{v_x}{v}\right\|^2 + \int_0^t\int_0^L \theta v_x^2\,dxds \\
&\quad\le C_1 + C_1\int_0^t\int_0^L \Big[|u\theta v_x| + |u\theta_x| + |\theta^3 u\theta_x| + |\theta_x v_x| \\
&\qquad + |\theta^3 v_x\theta_x| + \left|u^2\left(r^{1-n}u - \epsilon\frac{v_x}{v}\right)\right|\Big]dxds.
\end{aligned} \tag{2.3.46}$$

Noting the following facts,

$$\begin{aligned}
\int_0^t \|u(s)\|_{L^\infty}^2 ds &\le \int_0^t \left(\int_0^L \frac{u_x^2}{\theta}\right)\left(\int_0^L \theta\right)ds \\
&\le C_1\int_0^t\int_0^L \frac{u_x^2}{\theta}dxds \le C_1,
\end{aligned} \tag{2.3.47}$$

$$\int_0^L \theta^4 dx \le C_1, \tag{2.3.48}$$

$$\begin{aligned}\int_0^L \frac{v\theta^l}{\kappa(v,\theta)}dxds &\le C_1\int_0^L (1+\theta)^{l-q}dxds\\ &\le C_1 + C_1A^{\max\{l-q-4,0\}}, \quad l\ge 0, \end{aligned}\tag{2.3.49}$$

and using lemmas 2.3.1–2.3.3, we easily derive that for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^t\int_0^L |\theta u v_x|dxds &\le \varepsilon\int_0^t\int_0^L \theta v_x^2dxds + C_1(\varepsilon)\int_0^t\int_0^L \theta u^2dxds\\ &\le \varepsilon\int_0^t\int_0^L \theta v_x^2dxds + C_1(\varepsilon)\int_0^t \|u\|_{L^\infty}^2\int_0^L \theta dxds\\ &\le \varepsilon\int_0^t\int_0^L \theta v_x^2dxds + C_1(\varepsilon), \end{aligned}\tag{2.3.50}$$

$$\begin{aligned}\int_0^t\int_0^L |\theta_x u|dxds &\le \left(\int_0^t V_1(s)ds\right)^{\frac12}\left(\int_0^t\int_0^L \frac{v\theta^2u^2}{\kappa(v,\theta)}dxds\right)^{\frac12}\\ &\le C_1\left(\int_0^t \|u\|_{L^\infty}^2\left(\int_0^L \theta^2dx\right)ds\right)^{\frac12} \le C_1, \end{aligned}\tag{2.3.51}$$

$$\begin{aligned}\int_0^t\int_0^L |\theta^3\theta_x u|dxds &\le \left(\int_0^t V_1(s)ds\right)^{\frac12}\left(\int_0^t\int_0^L \frac{v\theta^8u^2}{\kappa(v,\theta)}dxds\right)^{\frac12}\\ &\le C_1 + C_1A^{\frac{q_0}{2}}, \end{aligned}\tag{2.3.52}$$

$$\begin{aligned}\int_0^t\int_0^L |\theta_x v_x|dxds &\le \left(\int_0^t V_1(s)ds\right)^{\frac12}\left(\int_0^t\int_0^L \frac{v\theta^2v_x^2}{\kappa(v,\theta)}dxds\right)^{\frac12}\\ &\le C_1\sup_{0\le s\le t}\left\|\frac{v\theta}{\kappa(v,\theta)}\right\|_{L^\infty}^{\frac12}\left(\int_0^t\int_0^L \theta v_x^2dxds\right)^{\frac12}\\ &\le \varepsilon\int_0^t\int_0^L \theta v_x^2dxds + C_1, \end{aligned}\tag{2.3.53}$$

$$\begin{aligned}&\int_0^t\int_0^L |(\theta^2-\tilde\theta^2)\theta\theta_x v_x|dxds\\ &\le C_1\int_0^t\int_0^L |\theta\theta_x|dx\int_0^L |\theta\theta_x v_x|dxds\\ &\le C_1\sup_{0\le s\le t}\left\{\left(\int_0^L \frac{v\theta^4}{\kappa(v,\theta)}dx\right)^{\frac12}\left(\int_0^L \frac{v\theta^4v_x^2}{\kappa(v,\theta)}dx\right)^{\frac12}\right\}\int_0^t V_1(s)ds\\ &\le \varepsilon\sup_{0\le s\le t}\|v_x(s)\|_{L^\infty}^2 + C_1, \end{aligned}\tag{2.3.54}$$

$$\begin{aligned}
&\int_0^t \int_0^L |\theta^3 \theta_x v_x| dxds \\
&\quad \le \int_0^t \int_0^L |\Big(\theta^2 - \tilde{\theta}^2\Big)\theta\theta_x v_x| dxds + \int_0^t \int_0^L \tilde{\theta}^2 |\theta\theta_x v_x| dxds \\
&\quad \le \varepsilon \sup_{0 \le s \le t} \|v_x(s)\|_{L^\infty}^2 + C_1 \\
&\qquad + C_1 \Big(\int_0^t V_1(s) ds\Big)^{\frac{1}{2}} \Big(\int_0^t \int_0^L \frac{v\theta v_x^2}{\kappa(v,\theta)} dxds\Big)^{\frac{1}{2}} \\
&\quad \le \varepsilon \sup_{0 \le s \le t} \|v_x(s)\|_{L^\infty}^2 + C_1 + C_1 \Big\|\frac{v}{\kappa(v,\theta)}\Big\|_{L^\infty}^{\frac{1}{2}} \Big(\int_0^L \theta v_x^2 dx\Big)^{\frac{1}{2}} \\
&\quad \le \varepsilon \sup_{0 \le s \le t} \|v_x(s)\|_{L^\infty}^2 + \varepsilon \int_0^t \int_0^L \theta v_x^2 dxds + C_1,
\end{aligned} \tag{2.3.55}$$

$$\begin{aligned}
&\int_0^t \int_0^L \Big|u^2 \Big(r^{1-n} u - \epsilon \frac{v_x}{v}\Big)\Big| dxds \\
&\quad \le C_1 \int_0^t \int_0^L u^2 dxds + C_1 \int_0^t \int_0^L u^2 \Big(r^{1-n} u - \epsilon \frac{v_x}{v}\Big)^2 dxds \\
&\quad \le C_1 \int_0^t \|u\|_{L^\infty}^2 ds + C_1 \int_0^t \|u\|_{L^\infty}^2 \int_0^L \Big(r^{1-n} u - \epsilon \frac{v_x}{v}\Big)^2 dxds \\
&\quad \le C_1 + C_1 \int_0^t \|u\|_{L^\infty}^2 \Big\|r^{1-n} u - \epsilon \frac{v_x}{v}\Big\|^2 ds.
\end{aligned} \tag{2.3.56}$$

On the other hand, it is easy to verify that

$$\|v_x\|^2 \le C \Big\|r^{1-n} u - \epsilon \frac{v_x}{v}\Big\|^2 + C_1. \tag{2.3.57}$$

Inserting (2.3.50)–(2.3.57) into (2.3.46), we arrive at

$$\Big\|r^{1-n} u - \epsilon \frac{v_x}{v}\Big\|^2 + \int_0^t \int_0^L \theta v_x^2 dxds \le C_1 + C_1 A^{\frac{q_0}{2}} + C_1 \int_0^t \|u\|_{L^\infty}^2 \Big\|r^{1-n} u - \epsilon \frac{v_x}{v}\Big\|^2 ds. \tag{2.3.58}$$

By the Bellman–Gronwall inequality, it follows from (2.3.47) and (2.3.58) that

$$\Big\|r^{1-n} u - \epsilon \frac{v_x}{v}\Big\|^2 + \int_0^t \int_0^L \theta v_x^2 dxds \le C_1 + C_1 A^{\frac{q_0}{2}}. \tag{2.3.59}$$

Moreover, noting that, from corollary 2.3.1,

$$1 \le C_1 \theta + C_1 V_1(t), \tag{2.3.60}$$

we derive from (2.3.59) and (2.3.60) that

$$\begin{aligned}\int_0^t\int_0^L v_x^2\,dxds &\le C_1\int_0^t\int_0^L \theta v_x^2\,dxds + C_1\int_0^t V_1(s)\|v_x(s)\|^2\,ds\\ &\le C_1 + C_1A^{\frac{q_0}{2}}\end{aligned}$$

which, together with (2.3.59), gives (2.3.44). □

The following lemma plays an important role in the subsequent estimates on the absolute temperature θ and the velocity u.

Lemma 2.3.5. *Under assumptions in theorem 2.2.1, the following estimates hold for any $t > 0$,*

$$\int_0^t\int_0^L (1+\theta)^{2m} v_x^2(x,s)\,dxds \le C_1 + C_1A^{\frac{q_0}{2}}, \tag{2.3.61}$$

$$\int_0^t\int_0^L (1+\theta)^{2m} u^2(x,s)\,dxds \le C_1 \tag{2.3.62}$$

with $0 \le m \le \frac{q+4}{2}$.

Proof. It follows from corollary 2.3.1 and lemma 2.3.4 that

$$\int_0^t\int_0^L (1+\theta)^{2m} v_x^2\,dxds \le C_1\int_0^t \|v_x(s)\|^2\,ds + \int_0^t V(s)\|v_x(s)\|^2\,ds \le C_1 + C_1A^{\frac{q_0}{2}}.$$

The proof of (2.3.62) is similar to that of (2.3.61). □

Now we turn to estimate the velocity u in H^1.

Lemma 2.3.6. *Under assumptions in theorem 2.2.1, the following estimates hold that for any $t > 0$,*

$$\|u(t)\|^2 + \int_0^t \|u_x(s)\|^2\,ds \le C_1 + C_1A^{q_0}, \tag{2.3.63}$$

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(s)\|^2\,ds \le C_1(1+A)^{q_1}, \tag{2.3.64}$$

$$\|u_x(t)\|^2 + \int_0^t \|u_t(s)\|^2\,ds \le C_1 + C_1A^{q_1} \tag{2.3.65}$$

where $q_1 = \max\{8-q, 0\}$.

Proof. Multiplying (2.2.28) by u and then integrating the resultants over Q_t, using the Young inequality and lemmas 2.3.1–2.3.5, we get

$$
\begin{aligned}
&\|u(t)\|^2+\int_0^t\|u_x(s)\|^2ds\\
&\quad\le C_1\int_0^t\int_0^L u^2dxds+C_1\int_0^t\int_0^L\left|(1+\theta^3)\theta_xu+\theta v_xu\right|dxds\\
&\quad\le C_1\int_0^t\|u\|^2_{L^\infty}ds+C_1\int_0^tV_1(s)ds+C_1\int_0^t\int_0^L\left(\theta v_x^2+\theta u^2+\frac{(1+\theta^3)^2\theta^2}{1+\theta^q}u^2\right)dxds\\
&\quad\le C_1+C_1A^{\frac{q_0}{2}}+C_1\int_0^t\|u(s)\|^2_{L^\infty}\left(\int_0^L\theta dx+(1+\|\theta\|_{L^\infty})^{q_0}\int_0^L(1+\theta^4)dx\right)ds\\
&\quad\le C_1+C_1A^{q_0}. \qquad (2.3.66)
\end{aligned}
$$

Multiplying (2.2.28) by u_{xx}, and then integrating the resultant over $[0,\ L]$ with respect to x, we have for any $\varepsilon>0$,

$$
\begin{aligned}
&\frac{d}{dt}\|u_x\|^2+\int_0^Lu_{xx}^2dx\\
&\quad\le C_1\int_0^L\left|u_xu_{xx}+uv_xu_{xx}+uu_{xx}+v_xu_xu_{xx}+(1+\theta^3)\theta_xu_{xx}+\theta v_xu_{xx}\right|dx\\
&\quad\le\varepsilon\int_0^Lu_{xx}^2dx+C_1(\varepsilon)\int_0^L\left(u_x^2+v_x^2+u^2+u^2v_x^2+u_x^2v_x^2+(1+\theta^3)^2\theta_x^2+\theta^2v_x^2\right)dx.
\end{aligned}
\qquad (2.3.67)
$$

Taking $\varepsilon>0$ small enough and integrating (2.3.67) over $[0,\ t]$ with respect to t, using lemmas 2.3.1–2.3.5 and (2.3.63), we can conclude

$$
\begin{aligned}
&\|u_x(t)\|^2+\int_0^t\|u_{xx}(s)\|^2ds\\
&\quad\le C_1+C_1A^{q_0}+C_1\left(1+A^{\frac{q_0}{2}}\right)\int_0^t\|u_x(s)\|^2_{L^\infty}ds+C_1(1+A)^{q_1}\int_0^tV_1(s)ds\\
&\quad\le C_1+C_1A^{q_0}+C_1\left(1+A^{\frac{q_0}{2}}\right)\left(\int_0^t\|u_x(s)\|^2ds\right)^{\frac12}\left(\int_0^t\|u_{xx}(s)\|^2ds\right)^{\frac12}+C_1(1+A)^{q_1}\\
&\quad\le C_1+C_1A^{2q_0}+C_1A^{q_1}+\frac12\int_0^t\|u_{xx}(s)\|^2ds,
\end{aligned}
$$

that is,

$$\|u_x(t)\|^2+\int_0^t\|u_{xx}(s)\|^2ds\le C_1(1+A)^{q_1},$$

with $q_1=\max\{8-q,0\}$.

Multiplying (2.2.28) by u_t, and then integrating the resultant over Q_t, using lemmas 2.3.1–2.3.5 and (2.3.63), (2.3.64), we can derive for any $\varepsilon>0$,

$$\begin{aligned}
&\|u_x(t)\|^2+\int_0^t\|u_t(s)\|^2ds\\
&\le C_1+C_1\left|\int_0^t\int_0^L\left[u_t\big(u_x+uv_x+u+u_xv_x+\big(1+\theta^3\big)\theta_x+\theta v_x\big)+u_x^2+u_x^3\right]dxds\right|\\
&\le C_1+\varepsilon\int_0^t\|u_t\|^2ds+C_1(\varepsilon)\int_0^t\int_0^L\Big(u_x^2+u^2v_x^2+u^2+u_x^2v_x^2+\big(1+\theta^3\big)^2\theta_x^2+\theta^2v_x^2+u_x^3\Big)dxds\\
&\le C_1+\varepsilon\int_0^t\|u_t\|^2ds+C_1A^{q_0}+C_1(1+A)^{q_1}\int_0^tV_1(s)ds+C_1\int_0^t\|u_x\|^{\frac{5}{2}}\|u_{xx}\|^{\frac{1}{2}}ds\\
&\le\varepsilon\int_0^t\|u_t\|^2ds+C_1(1+A)^{q_1}+C_1\sup_{0\le s\le t}\|u_x(s)\|\left(\int_0^t\|u_x(s)\|^2ds\right)^{\frac{3}{4}}\left(\int_0^t\|u_{xx}(s)\|^2ds\right)^{\frac{1}{4}}\\
&\le C_1+C_1A^{q_1}+\varepsilon\int_0^t\|u_t\|^2ds+C_1A^{\frac{3q_0+q_1}{2}}+\varepsilon\sup_{0\le s\le t}\|u_x(s)\|^2.
\end{aligned}$$

Thus taking $\varepsilon>0$ small enough, we have

$$\|u_x(t)\|^2+\int_0^t\|u_t(s)\|^2ds\le C_1+C_1A^{q_1}.$$

Thus the proof is complete. □

Corollary 2.3.2. *Under assumptions in theorem 2.2.1, the following estimate holds for any $t>0$,*

$$\int_0^t\int_0^L(1+\theta)^{2m}u_x^2dxds\le C_1(1+A)^{q_1}\tag{2.3.68}$$

where $0\le m\le\frac{q+4}{2}$.

Proof. It follows from corollary 2.3.1 and (2.3.63), (2.3.64) that

$$\begin{aligned}
\int_0^t\int_0^L(1+\theta)^{2m}u_x^2dxds&\le C_1\int_0^t\int_0^Lu_x^2dxds+C_1\int_0^t\int_0^LV_1(s)u_x^2dxds\\
&\le C_1+C_1A^{q_0}+C_1A^{q_1}\le C_1(1+A)^{q_1}.
\end{aligned}$$

□

Subsequently, we shall establish estimates on the absolute temperature θ by the following two lemmas.

Lemma 2.3.7. *Under assumptions in theorem 2.2.1, the following estimate holds that for any $t>0$,*

$$\left\|\theta(t)+\theta^4(t)\right\|^2+\int_0^t\int_0^L(1+\theta)^{q+3}\theta_x^2dxds\le C_1+C_1A^{q_2}\tag{2.3.69}$$

where

$$q_2 = \begin{cases} \max\left\{\frac{1}{2}q_0 + 1, \frac{q_0 + q_1}{2}\right\}, & \text{if} \quad 0 < \beta \le 8, \\ \max\left\{\frac{1}{2}q_0 + 1, \frac{q_0 + q_1}{2}, 4\right\}, & \text{if} \quad \beta > 8. \end{cases}$$

Proof. Multiplying (2.2.29) by e and integrating the resultant over Q_t, we have

$$\begin{aligned} &\left\|\theta(t) + \theta^4(t)\right\|^2 + \int_0^t \int_0^L (1+\theta)^{q+3}\theta_x^2 dxds \\ &\le C_1 + C_1 \int_0^t \int_0^L |(Pe)_x u + eu_x^2 + \kappa(v,\theta)v^{-1}\theta_x\theta^4 v_x + \lambda\phi Ze| dxds \\ &\le C_1 + C_1 \int_0^t \int_0^L [(1+\theta)^7|\theta_x u| + (1+\theta)^5|uv_x| + (1+\theta)^{q+4}|\theta_x v_x| \\ &\quad + (1+\theta)^8|uv_x| + eu_x^2 + |\phi Ze|] dxds. \end{aligned} \tag{2.3.70}$$

By lemmas 2.3.1–2.3.6, we derive for any $\varepsilon > 0$,

$$\begin{aligned} &\int_0^t \int_0^L (1+\theta)^7|\theta_x u| dxds \\ &\le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_x^2 dxds + C_1(\varepsilon) \int_0^t \int_0^L (1+\theta)^{11-q} u^2 dxds \\ &\le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_x^2 dxds + C_1 + C_1 A^{\max\{7-2q,0\}}, \end{aligned} \tag{2.3.71}$$

$$\begin{aligned} &\int_0^t \int_0^L (1+\theta)^8|v_x u| dxds \\ &\le C_1 \int_0^t \int_0^L (1+\theta)^8 v_x^2 dxds + C_1 \int_0^t \int_0^L (1+\theta)^8 u^2 dxds \\ &\le C_1 A^{q_0} \int_0^t \int_0^L (1+\theta)^{q+4} v_x^2 dxds + C_1 A^{q_0} \int_0^t \int_0^L (1+\theta)^{q+4} u^2 dxds \\ &\le C_1 + C_1 A^{\frac{3}{2}q_0}, \end{aligned} \tag{2.3.72}$$

$$\begin{aligned} &\int_0^t \int_0^L (1+\theta)^{q+4}|\theta_x v_x| dxds \\ &\le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_x^2 dxds + C_1(\varepsilon) \int_0^t \int_0^L (1+\theta)^{q+5} v_x^2 dxds \\ &\le C_1 + C_1 A^{\frac{1}{2}q_0+1} + \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_x^2 dxds, \end{aligned} \tag{2.3.73}$$

$$\begin{aligned}\int_0^t\int_0^L eu_x^2dxds &\le \int_0^t \|u_x(s)\|_{L^\infty}^2\left(\int_0^L edx\right)ds\\ &\le C_1\left(\int_0^t\|u_x(s)\|^2ds\right)^{\frac{1}{2}}\left(\int_0^t\|u_{xx}(s)\|^2ds\right)^{\frac{1}{2}}\\ &\le C_1+C_1A^{\frac{q_0+q_1}{2}}, \qquad (2.3.74)\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^L|\phi Ze|dxds &\le C_1\int_0^t\int_0^L(\theta+\theta^4)\phi Zdxds\\ &\le C_1\int_0^t\int_0^L\theta\phi Zdxds+C_1\int_0^t\int_0^L\theta^4\phi Zdxds\\ &\le C_1+C_1\int_0^t\int_0^L(1+V_1(s))\phi Zdxds\\ &\le C_1+C_1\int_0^t V_1(s)\int_0^L\phi dxds\\ &\le C_1+C_1\int_0^t V_1(s)\int_0^L\theta^\beta dxds\\ &\le C_1+C_1\int_0^t V_1(s)\int_0^L\theta^8 dxds, \quad \text{for } 0<\beta\le 8 \qquad (2.3.75)\end{aligned}$$

or

$$\begin{aligned}\int_0^t\int_0^L|\phi Ze|dxds &\le \sup_{0\le s\le t}\|e(s)\|_{L^\infty}\int_0^t\int_0^L\phi Zdxds\\ &\le C_1+C_1A^4, \quad \text{for } \beta>8. \qquad (2.3.76)\end{aligned}$$

Thus for $0<\beta\le 8$, we derive from (2.3.70)–(2.3.75) that

$$\begin{aligned}\int_0^L\theta^8dx+\int_0^t\int_0^L(1+\theta)^{q+3}\theta_x^2dxds &\le 2\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_x^2dxds+C_1\\ &\quad +C_1A^{1+\frac{1}{2}q_0}+C_1A^{\frac{q_0+q_1}{2}}+C_1\int_0^t V_1(s)\int_0^L\theta^8dxds\end{aligned}$$

which, by taking $\varepsilon>0$ sufficiently small and using the Gronwall inequality, gives (2.3.69). While for $\beta>8$, taking $\varepsilon>0$ small enough, we derive from (2.3.70)–(2.3.74) and (2.3.76) that

$$\int_0^L\theta^8dx+\int_0^t\int_0^L(1+\theta)^{q+3}\theta_x^2dxds\le C_1+C_1A^{1+\frac{1}{2}q_0}+C_1A^{\frac{q_0+q_1}{2}}+C_1A^4$$

which also implies the desired estimate (2.3.69). □

Lemma 2.3.8. *Under assumptions in theorem 2.2.1, the following estimate holds that for any $t>0$,*

$$\int_0^L(1+\theta)^{2q}\theta_x^2dx+\int_0^t\int_0^1(1+\theta)^{q+3}\theta_t^2dxds\le C_1(1+A)^{q_9}, \qquad (2.3.77)$$

where

$$\begin{cases} q_3 = \max\{q-3, 0\}, \\ q_4 = \max\left\{\dfrac{q-10}{2}, 0\right\}, \\ q_5 = \max\{1+q_1, 2q_1+q_4, q_3\}, \\ q_6 = \max\{3-q, 0\}, \\ q_7 = \max\left\{q_0+q_2+q_3+q_6, \dfrac{1}{4}(3q_0+3q_1+q_2+q_3), \dfrac{1}{2}(q_0+q_2+q_3+\beta)\right\}, \\ q_8 = \max\left\{q_3+\beta, \dfrac{1}{2}(q_1+\beta)\right\}, \\ q_9 = \max\left\{q_5, q_7, 1+\dfrac{3}{4}(q_0+q_1), \dfrac{1}{2}(q_1+q_2+q-1), q_3+\dfrac{1}{2}(q_0+q_1), q_8\right\}. \end{cases}$$

Proof. Let

$$H(x,t) = H(v,\theta) = \int_0^\theta \frac{r^{2n-2}\kappa(v,\xi)}{v} d\xi = \frac{\kappa_1 r^{2n-2}\theta}{v} + \frac{\kappa_2 r^{2n-2}\theta^{q+1}}{q+1}.$$

Then it is easy to verify that

$$H_t = H_v (r^{n-1}u)_x + \frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t,$$
$$H_{xt} = \left(\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}\right)_t + H_v (r^{n-1}u)_{xx} + H_{vv} v_x (r^{n-1}u)_x + \left(\frac{r^{2n-2}\kappa(v,\theta)}{v}\right)_v v_x\theta_t.$$

However, we easily know that

$$|H_v| + |H_{vv}| \le C(1+\theta), \quad \left|\left(\frac{r^{2n-2}\kappa(v,\theta)}{v}\right)_v\right| \le C_1. \tag{2.3.78}$$

We rewrite (2.2.29) as

$$e_\theta\theta_t + \theta P_\theta \left(r^{n-1}u\right)_x - \epsilon\frac{\left(r^{n-1}u\right)_x^2}{v} = \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x - 2\mu(n-1)\left(r^{n-2}u^2\right)_x + k\lambda\phi Z. \tag{2.3.79}$$

Multiplying (2.3.79) by H_t and integrating the resultant over Q_t, we easily obtain

$$\int_0^t\int_0^L \left(e_\theta\theta_t + \theta P_\theta\left(r^{n-1}u\right)_x - \epsilon\frac{\left(r^{n-1}u\right)_x^2}{v}\right) H_t dxds + \int_0^t\int_0^L \frac{r^{2n-2}\kappa\theta_x}{v} H_{tx} dxds$$
$$+ 2\mu(n-1)\int_0^t\int_0^L \left(r^{n-2}u^2\right)_x H_t dxds = \int_0^t\int_0^L k\lambda\phi Z H_t dxds. \tag{2.3.80}$$

Now we estimate each term in (2.3.80) by lemmas 2.3.1–2.3.7 and corollaries 2.3.1 and 2.3.2. Obviously,

$$\begin{aligned}\int_0^t\int_0^L e_\theta\theta_t\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t dxds &\geq C_1^{-1}\int_0^t\int_0^L(1+\theta^3)(1+\theta^q)\theta_t^2 dxds\\ &\geq C_1^{-1}\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds \end{aligned}\tag{2.3.81}$$

and noting the fact that

$$|\left(r^{n-1}u\right)_x|\leq C_1(|u|+|u_x|),\tag{2.3.82}$$

we have for any $\varepsilon>0$,

$$\begin{aligned}&\left|\int_0^t\int_0^L e_\theta\theta_t H_v\left(r^{n-1}u\right)_x dxds\right|\\ &\quad\leq C_1\int_0^t\int_0^L(1+\theta)^4|\theta_t|(|u|+|u_x|)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)\int_0^t\int_0^L(1+\theta)^{5-q}\left(u^2+u_x^2\right)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)\int_0^t\int_0^L(1+\theta)^{q+4}\left(u^2+u_x^2\right)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)(1+A)^{q_1}.\end{aligned}\tag{2.3.83}$$

Applying the Cauchy inequality and (2.3.82), we can derive for any $\varepsilon>0$,

$$\begin{aligned}&\left|\int_0^t\int_0^L\left(\theta P_\theta\left(r^{n-1}u\right)_x-\epsilon\frac{\left(r^{n-1}u\right)_x^2}{v}\right)\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t dxds\right|\\ &\quad\leq C_1\int_0^t\int_0^L\left((1+\theta)^{q+4}|\left(r^{n-1}u\right)_x\theta_t|+(1+\theta)^q|\left(r^{n-1}u\right)_x|^2|\theta_t|\right)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)\int_0^t\int_0^L\left((1+\theta)^{q+5}|\left(r^{n-1}u\right)_x|^2\right.\\ &\qquad\left.+(1+\theta)^{q-3}|\left(r^{n-1}u\right)_x|^4\right)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)\int_0^t\int_0^L\left((1+\theta)^{q+5}\left(|u|^2+|u_x|^2\right)\right.\\ &\qquad\left.+(1+\theta)^{q-3}\left(|u|^4+|u_x|^4\right)\right)dxds\\ &\quad\leq\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2 dxds+C_1(\varepsilon)(1+A)^{1+q_1}\\ &\qquad+C_1(\varepsilon)\int_0^t\int_0^L(1+\theta)^{q-3}\left(|u|^4+|u_x|^4\right)dxds.\end{aligned}\tag{2.3.84}$$

But

$$\begin{aligned}\int_0^t\int_0^L(1+\theta)^{q-3}u^4 dxds &\leq(1+A)^{q_3}\int_0^t\|u\|_{L^\infty}^2\int_0^L u^2 dxds\\ &\leq C_1(1+A)^{q_3},\end{aligned}\tag{2.3.85}$$

$$\begin{aligned}
&\int_0^t\int_0^L (1+\theta)^{q-3}u_x^4\,dxds\\
&\quad\le C_1(1+A)^{q_4}\int_0^t\int_0^L (1+\theta)^{\frac{q+4}{2}}u_x^4\,dxds\\
&\quad\le C_1(1+A)^{q_4}\left(\int_0^t\int_0^L u_x^4\,dxds+\int_0^t V_1^{\frac{1}{2}}(s)\int_0^L u_x^4\,dxds\right)\\
&\quad\le C_1(1+A)^{q_4}\left(\int_0^t \|u_x\|^3\|u_{xx}\|\,ds+\int_0^t V_1^{\frac{1}{2}}(s)\|u_x\|^3\|u_{xx}\|\,ds\right)\\
&\quad\le C_1(1+A)^{q_4}\left(\sup_{0\le s\le t}\|u_x(s)\|^2\left(\int_0^t\|u_x\|^2ds\right)^{\frac{1}{2}}\left(\int_0^t\|u_{xx}\|^2ds\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+\sup_{0\le s\le t}\|u_x(s)\|^3\left(\int_0^t V_1(s)\,ds\right)^{\frac{1}{2}}\left(\int_0^t\|u_{xx}\|^2ds\right)^{\frac{1}{2}}\right)\\
&\quad\le C_1(1+A)^{2q_1+q_4},
\end{aligned}\tag{2.3.86}$$

which, along with (2.3.84) and (2.3.85), gives

$$\begin{aligned}
&\int_0^t\int_0^L\left(\theta P_\theta\big(r^{n-1}u\big)_x-\epsilon\frac{\big(r^{n-1}u\big)_x^2}{v}\right)\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t\,dxds\\
&\quad\le\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)(1+A)^{q_5}
\end{aligned}\tag{2.3.87}$$

with $q_3=\max\{q-3,0\}$, $q_4=\max\{\frac{q-10}{2},0\}$, $q_5=\max\{1+q_1,2q_1+q_4,q_3\}$.

It follows from lemma 2.3.6 and corollary 2.3.2 that

$$\begin{aligned}
&\left|\int_0^t\int_0^L\left(\theta P_\theta\big(r^{n-1}u\big)_x-\epsilon\frac{\big(r^{n-1}u\big)_x^2}{v}\right)H_v\big(r^{n-1}u\big)_x\,dxds\right|\\
&\quad\le C_1\int_0^t\int_0^L\Big((1+\theta)^5\big|\big(r^{n-1}u\big)_x\big|^2+(1+\theta)\big|\big(r^{n-1}u\big)_x\big|^3\Big)dxds\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^{q+4}\big|\big(r^{n-1}u\big)_x\big|^2dxds\\
&\qquad+C_1(1+A)\int_0^t\int_0^L\Big(|u|^3+|u_x|^3\Big)dxds\\
&\quad\le C_1(1+A)^{q_1}+C_1(1+A)\int_0^t\|u\|_{L^\infty}^2\int_0^L|u|\,dxds\\
&\qquad+C_1(1+A)\int_0^t\int_0^L|u_x|^3\,dxds\\
&\quad\le C_1(1+A)^{q_1}+C_1(1+A)+C_1(1+A)\int_0^t\|u_x(s)\|^{\frac{5}{2}}\|u_{xx}(s)\|^{\frac{1}{2}}ds\\
&\quad\le C_1(1+A)^{q_1}+C_1(1+A)\sup_{0\le s\le t}\|u_x(s)\|\left(\int_0^t\|u_x(s)\|^2ds\right)^{\frac{3}{4}}\left(\int_0^t\|u_{xx}(s)\|^2ds\right)^{\frac{1}{4}}\\
&\quad\le C_1(1+A)^{q_1}+C_1(1+A)^{1+\frac{3(q_0+q_1)}{4}}.
\end{aligned}\tag{2.3.88}$$

Now let us consider the various contributions in the second integral of (2.3.80). By lemmas 2.3.1–2.3.7 and corollary 2.3.2, we have

$$\int_0^t \int_0^L \frac{r^{2n-2}\kappa\theta_x}{v}\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_t dxds \geq C_1^{-1}\int_0^L (1+\theta)^{2q}\theta_x^2 dx - C_1, \tag{2.3.89}$$

$$\begin{aligned}
&\left|\int_0^t \int_0^L \frac{r^{2n-2}\kappa\theta_x}{v} H_v\left(r^{n-1}u\right)_{xx} dxds\right| \\
&\quad\leq C_1 \int_0^t \int_0^L (1+\theta)^{1+q}|\theta_x\left(r^{n-1}u\right)_{xx}| dxds \\
&\quad\leq C_1\left(\int_0^t \int_0^L (1+\theta)^{3+q}\theta_x^2 dxds\right)^{\frac{1}{2}}\left(\int_0^t \int_0^L (1+\theta)^{q-1}|\left(r^{n-1}u\right)_{xx}|^2 dxds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{q_2}{2}}\left(\int_0^t \int_0^L (1+\theta)^{q-1}\left(|u|^2+|u_x|^2+|u_{xx}|^2\right) dxds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{q_2+q_1+q-1}{2}}
\end{aligned} \tag{2.3.90}$$

and

$$\begin{aligned}
&\left|\int_0^t \int_0^L \frac{r^{2n-2}\kappa\theta_x}{v} H_{vv}v_x\left(r^{n-1}u\right)_x dxds\right| \\
&\quad\leq C_1 \int_0^t \int_0^L (1+\theta)^{1+q}|\theta_x v_x\left(r^{n-1}u\right)_x| dxds \\
&\quad\leq C_1\left(\int_0^t \int_0^L (1+\theta)^{3+q}\theta_x^2 dxds\right)^{\frac{1}{2}}\left(\int_0^t \int_0^L (1+\theta)^{q-1}v_x^2|\left(r^{n-1}u\right)_x|^2 dxds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{q_2+q-1}{2}}\left(\int_0^t \int_0^L v_x^2\left(r^{n-1}u\right)_x^2 dxds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{q_2+q-1}{2}}\left(\int_0^t \int_0^L v_x^2u^2 dxds + \int_0^t \int_0^L v_x^2u_x^2 dxds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{q_2+q-1}{2}}\left(\int_0^t \|u\|_{L^\infty}^2\|v_x\|^2 ds + \int_0^t \|u_x\|_{L^\infty}^2\|v_x\|^2 ds\right)^{\frac{1}{2}} \\
&\quad\leq C_1(1+A)^{\frac{2q_2+2q-2+q_0}{4}}\left(\int_0^t \|u_x(s)\|^2 ds\right)^{\frac{1}{4}}\left(\int_0^t \|u_{xx}(s)\|^2 ds\right)^{\frac{1}{4}} \\
&\quad\leq C_1(1+A)^{\frac{q_0+q_2+q-1}{2}+\frac{q_1}{4}}.
\end{aligned} \tag{2.3.91}$$

Noting the following facts

$$\begin{aligned}
\int_0^t\int_0^L \left(r^{n-1}u\right)_x^4 dxds &\leq C_1\int_0^t\int_0^L \left(u^4+u_x^4\right)dxds\\
&\leq C_1+C_1\int_0^t \|u_x(s)\|^3\|u_{xx}(s)\|ds\\
&\leq C_1+C_1\sup_{0\leq s\leq t}\|u_x(s)\|^2\left(\int_0^t\|u_x(s)\|^2ds\right)^{\frac{1}{2}}\left(\int_0^t\|u_{xx}(s)\|^2ds\right)^{\frac{1}{2}}\\
&\leq C_1(1+A)^{\frac{3q_1+q_0}{2}},\\
\int_0^t\int_0^L \left(r^{n-2}u^2\right)_x^2 dxds &\leq C_1\int_0^t\int_0^L \left(u^4+u^2u_x^2\right)dxds\\
&\leq C_1+C_1\int_0^t\|u_x\|_{L^\infty}^2ds\\
&\leq C_1(1+A)^{\frac{q_1+q_0}{2}},\\
\int_0^t\int_0^L \phi^2Z^2dxds &\leq C_1A^\beta\int_0^t\int_0^L\phi Z^2dxds\\
&\leq C_1(1+A)^\beta
\end{aligned}$$

and using equation (2.2.29), we can obtain

$$\begin{aligned}
&\int_0^t\left\|\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x\right\|^2ds\\
&\leq C_1\int_0^t\left(\|e_\theta\theta_t\|^2+\left\|\theta P_\theta\left(r^{n-1}u\right)_x\right\|^2+\left\|\left(r^{n-1}u\right)_x^2\right\|^2+\left\|\left(r^{n-2}u^2\right)_x\right\|^2+\|\phi Z\|^2\right)ds\\
&\leq C_1\int_0^t\int_0^L\left((1+\theta)^6\theta_t^2+(1+\theta)^8u^2+(1+\theta)^8u_x^2+\left(r^{n-1}u\right)_x^4+\left(r^{n-2}u^2\right)_x^2+\phi^2Z^2\right)dxds\\
&\leq C_1(1+A)^{q_6}\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2dxds+C_1(1+A)^{q_0}\int_0^t\int_0^L(1+\theta)^{q+4}u_x^2dxds\\
&\quad+C_1(1+A)^{\frac{3q_1+q_0}{2}}+C_1(1+A)^{\frac{q_1+q_0}{2}}+C_1(1+A)^\beta\\
&\leq C_1(1+A)^{q_6}\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2dxds+C_1(1+A)^{\frac{3q_1+q_0}{2}}+C_1(1+A)^\beta
\end{aligned}\tag{2.3.92}$$

with $q_6=\max\{3-q,0\}$. Thus, by the Sobolev inequality, lemmas 2.3.4 and 2.3.7, we can conclude for any $\varepsilon>0$,

$$
\begin{aligned}
&\left|\int_0^t\int_0^L \frac{r^{2n-2}\kappa\theta_x}{v}\left(\frac{r^{2n-2}\kappa}{v}\right)_v v_x\theta_t\,dxds\right|\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^q|\theta_x v_x\theta_t|\,dxds\\
&\quad\le \varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)\int_0^t\int_0^L\frac{(1+\theta)^{q-3}}{\kappa^2(v,\theta)}\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)^2 v_x^2\,dxds\\
&\quad\le \varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)\int_0^t\left\|\frac{r^{2n-2}\kappa\theta_x}{v}\right\|_{L^\infty}^2\|v_x\|^2\,ds\\
&\quad\le \varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)(1+A)^{\frac{q_0}{2}}\int_0^t\left\|\frac{r^{2n-2}\kappa\theta_x}{v}\right\|\left\|\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x\right\|ds\\
&\quad\le \varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)(1+A)^{\frac{q_0}{2}}\left(\int_0^t\left\|\frac{r^{2n-2}\kappa\theta_x}{v}\right\|^2ds\right)^{\frac12}\\
&\qquad\times\left(\int_0^t\left\|\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x\right\|^2ds\right)^{\frac12}\\
&\quad\le \varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)(1+A)^{\frac{q_0+q_3}{2}}\\
&\qquad\times\left(\int_0^t\int_0^L(1+\theta)^{q+3}\theta_x^2\,dxds\right)^{\frac12}\left(\int_0^t\left\|\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x\right\|^2ds\right)^{\frac12}\\
&\quad\le 2\varepsilon\int_0^t\int_0^L(1+\theta)^{q+3}\theta_t^2\,dxds+C_1(\varepsilon)(1+A)^{q_7}
\end{aligned}
\tag{2.3.93}
$$

where $q_7=\max\left\{q_0+q_2+q_3+q_6,\frac14(3q_0+3q_1+q_2+q_3),\frac12(q_0+q_2+q_3+\beta)\right\}$.

By lemma 2.3.5 and corollary 2.3.2, using the Hölder inequality, we have for any $\varepsilon>0$,

$$
\begin{aligned}
&\left|\int_0^t\int_0^L\left(r^{n-2}u^2\right)_x H_v\left(r^{n-1}u\right)_x\,dxds\right|\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)|u^3+uu_x^2|\,dxds\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^{\frac{q+4}{2}}|u|u^2\,dxds+C_1\int_0^t\int_0^L(1+\theta)^{\frac{q+4}{2}}|u|u_x^2\,dxds\\
&\quad\le C_1\left(\int_0^t\int_0^L(1+\theta)^{q+4}u^2\,dxds\right)^{\frac12}\\
&\qquad\times\left(\left(\int_0^t\int_0^L u^4\,dxds\right)^{\frac12}+\left(\int_0^t\int_0^L u_x^4\,dxds\right)^{\frac12}\right)\\
&\quad\le C_1(1+A)^{\frac{3q_1+q_0}{4}}
\end{aligned}
\tag{2.3.94}
$$

and

$$\begin{aligned}
&\left|\int_0^t \int_0^L \left(r^{n-2}u^2\right)_x \frac{r^{2n-2}\kappa\theta_t}{v}\,dxds\right| \\
&\quad \le C_1 \int_0^t \int_0^L (1+\theta)^q |u(u+u_x)||\theta_t|\,dxds \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)\int_0^t \int_0^L (1+\theta)^{q-3}u^2|u+u_x|^2\,dxds \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)(1+A)^{q_3} \\
&\qquad + C_1(\varepsilon)\int_0^t \int_0^L (1+\theta)^{q-3}u^2u_x^2\,dxds \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)(1+A)^{q_3+\frac{q_1+q_0}{2}}.
\end{aligned} \tag{2.3.95}$$

The term on the right-hand side of (2.3.80) can be estimated as follows, for any $\varepsilon > 0$,

$$\begin{aligned}
&\left|\int_0^t \int_0^L k\lambda\phi Z\left(H_v\left(r^{n-1}u\right)_x + \frac{r^{2n-2}\kappa}{v}\theta_t\right)dxds\right| \\
&\quad \le C_1 \int_0^t \int_0^L \left((1+\theta)\left|\phi Z\left(r^{n-1}u\right)_x\right| + (1+\theta)^q|\phi Z\theta_t|\right)dxds \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)\int_0^t \int_0^L (1+\theta)^{q-3}\phi^2Z^2\,dxds \\
&\qquad + C_1\left(\int_0^t \int_0^L \phi^2Z^2\,dxds\right)^{\frac{1}{2}}\left(\int_0^t \int_0^L (1+\theta)^2\left(r^{n-1}u\right)_x^2\,dxds\right)^{\frac{1}{2}} \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)(1+A)^{q_3+\beta} + C_1(1+A)^{\frac{q_1+\beta}{2}} \\
&\quad \le \varepsilon \int_0^t \int_0^L (1+\theta)^{q+3}\theta_t^2\,dxds + C_1(\varepsilon)(1+A)^{q_8}
\end{aligned} \tag{2.3.96}$$

with $q_8 = \max\left\{q_3+\beta, \frac{q_1+\beta}{2}\right\}$.

Therefore, taking $\varepsilon > 0$ small enough, we can obtain (2.3.77) and complete the proof from (2.3.80)–(2.3.96). □

Based on lemmas 2.3.1–2.3.8, we easily obtain the uniform upper bound of $\|\theta(t)\|_{L^\infty}$, which is the most key estimate in the proof of existence of global solution.

Lemma 2.3.9. *Under assumptions in theorem 2.2.1, the following estimate holds that for any $t > 0$,*

$$\|\theta(t)\|_{L^\infty} \le C_1. \tag{2.3.97}$$

Proof. Using the Sobolev embedding theorem, lemmas 2.3.1–2.3.8 and the Young inequality, we can derive

$$\begin{aligned}\left\|\theta^{q+5}(t)-\left(\int_0^L \theta\,dx\right)^{q+5}\right\|_{L^\infty} &\le C_1\int_0^L |\theta^{q+4}\theta_x|\,dx\\ &\le C_1\left(\int_0^L \theta^{2q}\theta_x^2\,dx\right)^{1/2}\left(\int_0^L \theta^8\,dx\right)^{1/2}\\ &\le C_1(1+A)^{\frac{q_9+q_2}{2}}\end{aligned}$$

which gives

$$A^{q+5}\le C_1+C_1A^{\frac{q_9+q_2}{2}}\le C_1+\frac{1}{2}A^{q+5} \tag{2.3.98}$$

where we have used that $(\beta, q)\in E$ implies $q_9+q_2<2q+10$. Therefore, we derive (2.3.97) from (2.3.98) and the Young inequality. □

The next lemma means the global existence of solutions in $\mathcal{H}^1$.

Lemma 2.3.10. *Under assumptions in theorem 2.2.1, the following estimate holds that for any* $t>0$*,*

$$\begin{aligned}&\|v_x(t)\|^2+\|u_x(t)\|^2+\|\theta_x(t)\|^2+\|Z_x(t)\|^2+\int_0^t(\|v_x\|^2+\|u_x\|^2\\ &\quad+\|\theta_x\|^2+\|Z_x\|^2+\|u_t\|^2+\|\theta_t\|^2+\|Z_t\|^2+\|u_{xx}\|^2+\|Z_{xx}\|^2)(s)\,ds\le C_1.\end{aligned} \tag{2.3.99}$$

Proof. Applying lemmas 2.3.4–2.3.9, we have

$$\begin{aligned}&\|v_x(t)\|^2+\|u_x(t)\|^2+\|\theta_x(t)\|^2+\int_0^t(\|v_x\|^2+\|u_x\|^2\\ &\quad+\|\theta_x\|^2+\|u_t\|^2+\|\theta_t\|^2+\|u_{xx}\|^2)(s)\,ds\le C_1.\end{aligned} \tag{2.3.100}$$

Multiplying (2.2.30) by Z_{xx} and then integrating the resultant over Q_t, using (2.3.100), lemmas 2.3.1 and 2.3.3, we can conclude

$$\begin{aligned}\|Z_x(t)\|^2+\int_0^t\|Z_{xx}(s)\|^2ds &\le C_1+C_1\int_0^t\int_0^L|Z_x+Z_xv_x||Z_{xx}|\,dxds+C_1\int_0^t\int_0^L|\phi Z||Z_{xx}|\,dxds\\ &\le C_1+C_1\int_0^t\int_0^L\left(Z_x^2+Z_x^2v_x^2+\phi^2Z^2\right)dxds+\frac{1}{4}\int_0^t\|Z_{xx}(s)\|^2ds\\ &\le C_1+C_1\int_0^t\|Z_x(s)\|_{L^\infty}^2ds+\frac{1}{4}\int_0^t\|Z_{xx}(s)\|^2ds\\ &\le C_1+C_1\int_0^t\|Z_x(s)\|\|Z_{xx}(s)\|\,ds+\frac{1}{4}\int_0^t\|Z_{xx}(s)\|^2ds\\ &\le C_1+\frac{1}{2}\int_0^t\|Z_{xx}(s)\|^2ds,\end{aligned}$$

that is,

$$\|Z_x(t)\|^2 + \int_0^t \|Z_{xx}(s)\|^2 ds \leq C_1. \tag{2.3.101}$$

Multiplying (2.2.30) by Z_t and then integrating the resultant over Q_t, using (2.3.100), lemmas 2.3.1, 2.3.3, 2.3.5 and (2.3.101), we can get

$$\begin{aligned}\|Z_x(t)\|^2 + \int_0^t \|Z_t(s)\|^2 ds &\leq C_1 + C_1 \int_0^t \int_0^L (u^2 + (r^{n-1}u)_x^2 + Z_x^4 + \phi^2 Z^2)\,dxds \\ &\leq C_1 + C_1 \int_0^t \|Z_x(s)\|_{L^\infty}^2 \|Z_x(s)\|^2 ds \\ &\leq C_1 + C_1 \int_0^t \|Z_x(s)\|^2 ds + C_1 \int_0^t \|Z_{xx}(s)\|^2 ds \\ &\leq C_1\end{aligned}$$

which, along with (2.3.100) and (2.3.101), leads to (2.3.99). □

2.3.2 Exponential Stability of Solutions in $\mathcal{H}^1$

In this subsection, we shall establish the exponential stability of solutions in $\mathcal{H}^1$. First, we show the asymptotic behavior of solutions in $\mathcal{H}^1$, which implies the uniqueness of solutions.

The next two lemmas are aimed at showing the asymptotic behavior of solutions to the problem (2.2.27)–(2.2.32) in $\mathcal{H}^1$.

Lemma 2.3.11. *Under assumptions in theorem 2.2.1, the following estimates are valid for any $t > 0$,*

$$\frac{d}{dt}\|v_x(t)\|^2 \leq C_1(\|u(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 + \|u_{xx}(t)\|^2), \tag{2.3.102}$$

$$\begin{aligned}\frac{d}{dt}\|u_x(t)\|^2 \leq C_1(&\|u(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \\ &+ \|u_{xx}(t)\|^2 + \|\theta_x(t)\|^2),\end{aligned} \tag{2.3.103}$$

$$\begin{aligned}\frac{d}{dt}\|\theta_x(t)\|^2 + \int_0^L (1+\theta)^{q-3}\theta_{xx}^2 dx \leq C_1(&\|u(t)\|^2 + \|u_x(t)\|^2 + \|u_{xx}(t)\|^2 \\ &+ \|\theta_x(t)\|^2 + \|\phi Z\|^2),\end{aligned} \tag{2.3.104}$$

$$\|\theta_x(t)\|^2 + \int_0^t \int_0^L (1+\theta)^{q-3}\theta_{xx}^2 dxds \le C_1, \tag{2.3.105}$$

$$\frac{d}{dt}\|Z_x(t)\|^2 \le C_1\Big(\|Z_x(t)\|^2 + \|Z_{xx}(t)\|^2 + \|\phi Z\|^2\Big). \tag{2.3.106}$$

Proof. Differentiating (2.2.27) with respect to x and multiplying the resultant by v_x yields the estimate (2.3.102).

Similarly, multiplying (2.2.28) and (2.2.30) by u_{xx}, Z_{xx}, respectively, and then integrating the resulting equations on $[0, L]$, using the Sobolev embedding theorem, we can derive estimates (2.3.103) and (2.3.106).

Multiplying (2.2.29) by $e_\theta^{-1}\theta_{xx}$ and integrating the resultant on $[0, L]$, we can derive from the Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned}
&\frac{d}{dt}\|\theta_x\|^2 + 2\int_0^L \frac{r^{2n-2}\kappa}{v}\theta_{xx}^2 e_\theta^{-1}\,dx \\
&\quad = \int_0^L \Bigg(\frac{\theta P_\theta (r^{n-1}u)_x}{e_\theta} - \epsilon\frac{(r^{n-1}u)_x^2}{ve_\theta} - \frac{(r^{2n-2}\kappa)_x\theta_x}{ve_\theta} \\
&\qquad -k\lambda\frac{\phi Z}{e_\theta} + \frac{r^{2n-2}\kappa\theta_x v_x}{v^2 e_\theta} + 2\mu(n-1)\frac{(r^{n-2}u^2)_x}{e_\theta}\Bigg)\theta_{xx}\,dx \\
&\quad \le \varepsilon\|\theta_{xx}\|^2 + C_1\Big(\|u\|^2 + \|u_x\|^2 + \|u_x\|_{L^4}^4 + \|\theta_x\|_{L^4}^4 + \|v_x\theta_x\|^2 + \|\phi Z\|^2\Big) \\
&\quad \le \varepsilon\|\theta_{xx}\|^2 + C_1(\|u\|^2 + \|u_x\|^2 + \|u_x\|^3\|u_{xx}\| \\
&\qquad + \|\theta_x\|^3\|\theta_{xx}\| + \|\theta_x\|^4 + \|\theta_x\|_{L^\infty}^2 + \|\phi Z\|^2) \\
&\quad \le 2\varepsilon\|\theta_{xx}\|^2 + C_1\Big(\|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|\theta_x\|^2 + \|\phi Z\|^2\Big).
\end{aligned}$$

Hence taking $\varepsilon > 0$, sufficiently small in the above inequality, we obtain (2.3.104) and (2.3.105). □

Lemma 2.3.12. *Under assumptions in theorem 2.2.1, we have, as $t \to +\infty$,*

$$\|v(t) - \overline{v}\|_{H^1} \to 0, \|u(t)\| \to 0, \|u(t)\|_{L^\infty} \to 0, \tag{2.3.107}$$

$$\|\theta_x(t)\| \to 0, \big\|\theta(t) - \overline{\theta}\big\|_{H^1} \to 0, \big\|\theta(t) - \overline{\theta}\big\|_{L^\infty} \to 0, \tag{2.3.108}$$

$$\|Z(t)\|_{H^1} \to 0, \tag{2.3.109}$$

and for all $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \le \theta(x, t) \le C_1. \tag{2.3.110}$$

Proof. By lemmas 2.3.1–2.3.11, we can know

$$\int_0^t \left(\|v_x\|^2 + \|u_x\|^2 + \|\theta_x\|^2 + \|Z_x\|^2\right)(s)\,ds \le C_1 \tag{2.3.111}$$

and

$$\int_0^t \left(\left|\frac{d}{dt}\|v_x\|^2\right| + \left|\frac{d}{dt}\|u_x\|^2\right| + \left|\frac{d}{dt}\|\theta_x\|^2\right| + \left|\frac{d}{dt}\|Z_x\|^2\right|\right)(s)\,ds \le C_1 \tag{2.3.112}$$

which lead to (2.3.107)–(2.3.109).

We derive from (2.3.108) that there exists a large time $t_0 > 0$ such that for any $t \geq t_0$,

$$\theta(x,t) \ge \frac{1}{2}\overline{\theta} > 0. \tag{2.3.113}$$

On the other hand, if we put $\Theta := \frac{1}{\theta}$, using $n\lambda_1 + 2\mu > 0$, and then taking $\delta > 0$ such that if $n = 2, 3$,

$$0 \le \frac{2(n-2)\mu}{(n-1)\epsilon} < \frac{2(n-1)\mu}{n\epsilon} < \delta < 1,$$

then (2.3.79) becomes

$$\begin{aligned} e_\theta \Theta_t = {} & \left(\frac{r^{2n-2}\kappa\Theta_x}{v}\right)_x - \left\{\frac{2\kappa r^{2n-2}\Theta_x^2}{v\Theta} + \frac{\Theta^2}{v}\left((n-1)\delta\epsilon - 2(n-2)\mu\right)\right. \\ & \times \left(\frac{uv}{r} + \frac{(\epsilon\delta - 2\mu)r^{n-1}u_x}{(n-1)\delta\epsilon - 2(n-2)\mu}\right)^2 + \frac{2\mu(\epsilon\delta\epsilon - 2(n-1)\mu)r^{2n-2}u_x^2\Theta^2}{v((n-1)\delta\epsilon - 2(n-2)\mu)} \\ & \left. + \epsilon(1-\delta)\frac{\Theta^2}{v}\left(\left(r^{n-1}u\right)_x - \frac{vP_\theta}{2\epsilon(1-\delta)\Theta}\right)^2 + k\lambda\phi Z\Theta^2\right\} + \frac{vP_\theta^2}{4(1-\delta)\epsilon}, \end{aligned} \tag{2.3.114}$$

which, together with lemmas 2.3.3 and 2.3.9, implies that there exists a positive constant C_1 such that

$$\Theta_t \le \frac{1}{e_\theta}\left(\frac{r^{2n-2}\kappa\Theta_x}{v}\right)_x + C_1.$$

Defining $\tilde{\Theta}(x,t) = C_1 t + \max_{x\in[0,L]}\frac{1}{\theta_0(x)} - \Theta(x,t)$ and a parabolic operator $\mathcal{L} := -\frac{\partial}{\partial t} + \frac{1}{e_\theta}\frac{\partial}{\partial x}\left(\frac{r^{2n-2}\kappa}{v}\frac{\partial}{\partial x}\right)$, we have a system

$$\begin{cases} \mathcal{L}\tilde{\Theta} \le 0, & \text{on } Q_{t_0}, \\ \tilde{\Theta}\big|_{t=0} \ge 0, & \text{on } [0,L], \\ \tilde{\Theta}_x\big|_{x=0,L} = 0, & \text{on } [0,t_0]. \end{cases}$$

Thus the standard comparison argument implies

$$\min_{(x,t)\in\overline{Q}_{t_0}} \tilde{\Theta}(x,t) \geq 0$$

which yields for any $(x, t) \in \overline{Q}_{t_0}$,

$$\theta(x,t) \geq \left(C_1 t + \max_{x\in[0,L]} \frac{1}{\theta_0(x)} \right)^{-1}.$$

Thus, for all $0 \leq t \leq t_0$,

$$\theta(x,t) \geq \left(C_1 t_0 + \max_{x\in[0,L]} \frac{1}{\theta_0(x)} \right)^{-1} \geq C_1^{-1}$$

which, along with (2.3.113) and lemma 2.3.9, gives (2.3.110). □

In the following lemmas, we shall establish the exponential stability of solutions in $\mathcal{H}^1$ using Qin's idea from [99]. Now we introduce the density of the fluids, $\rho = \frac{1}{v}$, then the specific entropy

$$\eta = \eta(\rho,\theta) = R\log v + C_v \log\theta + \frac{4a}{3} v\theta^3$$

satisfies

$$\frac{\partial \eta}{\partial \rho} = -\frac{P_\theta}{\rho^2}, \ \frac{\partial \eta}{\partial \theta} = \frac{e_\theta}{\theta}, \tag{2.3.115}$$

where $P = R\rho\theta + \frac{a}{3}\theta^4$. Considering the transform

$$\mathcal{A} : (\rho,\theta) \in \mathcal{D}_{\rho,\theta} = \{(\rho,\theta) : \rho > 0, \theta > 0\} \to (v,\eta) \in \mathcal{A}\mathcal{D}_{\rho,\theta},$$

and owing to the Jacobian $\left|\frac{\partial(v,\eta)}{\partial(\rho,\theta)}\right| = -\frac{e_\theta}{\rho^2\theta} = -\frac{1}{\rho^2}(C_v\theta^{-1} + 4a\rho^{-1}\theta^2) < 0$ on $\mathcal{D}_{\rho,\theta}$, we know that there is a unique function $\theta = \theta(v, \eta)$ as the smooth function of $(v,\eta) \in \mathcal{A}\mathcal{D}_{\rho,\theta}$. We denote by

$$e = e(v,\eta) := e(v,\theta(v,\eta)) = e\left(\frac{1}{\rho},\theta\right), \ P = P(v,\eta) := P(v,\theta(v,\eta)) = P\left(\frac{1}{\rho},\theta\right), \tag{2.3.116}$$

then we have

$$\begin{aligned} & e_v = -P, \ e_\eta = \theta, \\ & P_v = -\left(\rho^2 P_\rho + \frac{\theta P_\theta^2}{e_\theta}\right), \\ & P_\eta = \frac{\theta P_\theta}{e_\theta}, \\ & \theta_v = -\frac{\theta P_\theta}{e_\theta}, \ \theta_\eta = \frac{\theta}{e_\theta}. \end{aligned} \tag{2.3.117}$$

Define the following energy form $\mathcal{E}(v,u,\eta,Z)$ by

$$\begin{aligned}\mathcal{E}(v,u,\eta,Z) = &\frac{1}{2}u^2 + \lambda Z + e(v,\eta) - e(\overline{v},\overline{\eta})\\ &- \frac{\partial e}{\partial v}(\overline{v},\overline{\eta})(v-\overline{v}) - \frac{\partial e}{\partial \eta}(\overline{v},\overline{\eta})(\eta-\overline{\eta})\end{aligned} \tag{2.3.118}$$

where $e(v,\eta) = e(v,\theta) = C_v\theta + a\frac{\theta^4}{\rho}, \overline{v} = L^{-1}\int_0^L v_0(x)\,dx$ and $\overline{\theta} > 0$ is determined by $e\big(\overline{v},\overline{\theta}\big) = e(\overline{v},\overline{\eta}) = \int_0^L \left(\frac{u_0^2}{2} + e(v_0,\theta_0) + \lambda Z_0\right)dx$ and $\overline{\eta} = \eta(\overline{v},\overline{\theta})$.

The next lemma is aimed at establishing the bound of the energy $\mathcal{E}(v,u,\eta,Z)$.

Lemma 2.3.13. *Under assumptions in theorem 2.2.1, the unique generalized global solution $(v(t), u(t), \theta(t), Z(t))$ to the problem (2.2.27)–(2.2.32) satisfies the following estimates*

$$\begin{aligned}&\frac{1}{2}u^2 + \lambda Z + C_1^{-1}(|v-\overline{v}|^2 + |\eta-\overline{\eta}|^2)\\ &\quad\le \mathcal{E}(v,u,\eta,Z)\\ &\quad\le \frac{1}{2}u^2 + \lambda Z + C_1(|v-\overline{v}|^2 + |\eta-\overline{\eta}|^2).\end{aligned} \tag{2.3.119}$$

Proof. By the mean value problem theorem, there exists a point $(\tilde{v},\tilde{\eta})$ between (v,η) and $(\overline{v},\overline{\eta})$ such that

$$\begin{aligned}e(v,\eta) = {}& e(\overline{v},\overline{\eta}) + \frac{\partial e}{\partial v}(\overline{v},\overline{\eta})(v-\overline{v}) + \frac{\partial e}{\partial \eta}(\overline{v},\overline{\eta})(\eta-\overline{\eta})\\ &+ \frac{1}{2}\Big(\frac{\partial^2 e}{\partial v^2}(\tilde{v},\tilde{\eta})(v-\overline{v})^2 + 2\frac{\partial^2 e}{\partial v\partial\eta}(\tilde{v},\tilde{\eta})(v-\overline{v})(\eta-\overline{\eta})\\ &+ \frac{\partial^2 e}{\partial \eta^2}(\tilde{v},\tilde{\eta})(\eta-\overline{\eta})^2\Big).\end{aligned} \tag{2.3.120}$$

By (2.3.118) and (2.3.120), we get

$$\begin{aligned}\mathcal{E}(v,u,\eta,Z) = {}&\frac{1}{2}u^2 + \lambda Z + \frac{1}{2}\Big(\frac{\partial^2 e}{\partial v^2}(\tilde{v},\tilde{\eta})(v-\overline{v})^2\\ &+ 2\frac{\partial^2 e}{\partial v\partial\eta}(\tilde{v},\tilde{\eta})(v-\overline{v})(\eta-\overline{\eta}) + \frac{\partial^2 e}{\partial \eta^2}(\tilde{v},\tilde{\eta})(\eta-\overline{\eta})^2\Big),\end{aligned} \tag{2.3.121}$$

where

$$\tilde{v} = \lambda_0\overline{v} + (1-\lambda_0)v,\ \tilde{\eta} = \lambda_0\overline{\eta} + (1-\lambda_0)\eta,\ 0 \le \lambda_0 \le 1.$$

By lemma 2.3.3 and (2.3.110), we get

$$0 < C_1^{-1} \le \left|\tilde{v}\left(\frac{1}{\tilde{\rho}}, \tilde{\theta}\right)\right| \le C_1, \quad 0 < C_1^{-1} \le \left|\tilde{\eta}\left(\frac{1}{\tilde{\rho}}, \tilde{\theta}\right)\right| \le C_1,$$

which implies

$$\left|\frac{\partial^2 e}{\partial v^2}(\tilde{v}, \tilde{\eta})\right| + \left|\frac{\partial^2 e}{\partial v \partial \eta}(\tilde{v}, \tilde{\eta})\right| + \left|\frac{\partial^2 e}{\partial \eta^2}(\tilde{v}, \tilde{\eta})\right| \le C_1. \tag{2.3.122}$$

Thus, using the Cauchy inequality and (2.3.122), we have

$$\mathcal{E}(v, u, \eta, Z) \le \frac{1}{2}u^2 + \lambda Z + C_1\Big((v - \overline{v})^2 + (\eta - \overline{\eta})^2\Big). \tag{2.3.123}$$

On the other hand, we derive from (2.3.117) that

$$e_{vv} = -P_v = \rho^2 P_\rho + \frac{\theta P_\theta^2}{e_\theta}, \ e_{v\eta} = -P_\eta = \theta_v = -\frac{\theta P_\theta}{e_\theta}, \ e_{\eta\eta} = \theta_\eta = \frac{\theta}{e_\theta},$$

which yields that the Hessian of $e(v, \eta)$ is positive definite for any $v > 0$ and $\theta > 0$. Thus we derive from (2.3.120) that

$$\begin{aligned}\mathcal{E}(v, u, \eta, Z) &\ge \frac{1}{2}u^2 + \lambda Z + \lambda_{\min}(\tilde{v}, \tilde{\eta})\Big((v - \overline{v})^2 + (\eta - \overline{\eta})^2\Big)\\ &\ge \frac{1}{2}u^2 + \lambda Z + C_1^{-1}\Big((v - \overline{v})^2 + (\eta - \overline{\eta})^2\Big),\end{aligned} \tag{2.3.124}$$

where $\lambda_{\min}(\tilde{v}, \tilde{\eta}) \ \big(\ge C_1^{-1}\big)$ is the smaller characteristic root of the Hessian of $e(\tilde{v}, \tilde{\eta})$. Therefore, combining (2.3.123) and (2.3.124) yields (2.3.119). ☐

The following two lemmas concern with the exponential stability of the generalized global solutions $(v(t), u(t), \theta(t), Z(t))$ in $\mathcal{H}^1$.

Lemma 2.3.14. *Under assumptions in theorem 2.2.1, there exists a positive constant* $\gamma_1' = \gamma_1'(C_1) > 0$ *such that for any fixed* $\gamma \in (0, \gamma_1']$, *the generalized global solution* $(u(t), v(t), \theta(t), Z(t))$ *to the problem (2.2.27)–(2.2.32) satisfies the following estimate, for any* $t > 0$,

$$\begin{aligned}&e^{\gamma t}\Big(\|u(t)\|^2 + \|v(t) - \overline{v}\|^2 + \big\|\theta(t) - \overline{\theta}\big\|^2 + \|v_x(t)\|^2 + \|\rho_x(t)\|^2\\ &\quad + \|Z(t)\|^2 + \|Z(t)\|_{L^1(0,L)}\Big)\\ &\quad + \int_0^t e^{\gamma s}\left(\|u_x\|^2 + \|\rho_x\|^2 + \|\theta_x\|^2 + \|v_x\|^2 + \|Z_x\|^2 + \int_0^L \phi Z dx\right)(s)\, ds\\ &\quad\le C_1 + C_1\int_0^t e^{\gamma s}\|u(s)\|_{L^\infty}^2\|\theta_x(s)\|^2 ds.\end{aligned} \tag{2.3.125}$$

Proof. From (2.2.27)–(2.2.30), it is easy to verify that $(\rho,\, u,\, \eta)$ satisfies

$$\begin{aligned}\left(e+\frac{1}{2}u^2+\lambda Z\right)_t=\Bigg(&\frac{r^{2n-2}\kappa\theta_x}{v}+\left(\epsilon\frac{(r^{n-1}u)_x}{v}-P\right)r^{n-1}u\\&-2\mu(n-1)r^{n-2}u^2+\lambda d\frac{r^{2n-2}Z_x}{v^2}\Bigg)_x,\end{aligned}\tag{2.3.126}$$

$$\begin{aligned}\eta_t=&\left(\frac{\kappa\rho r^{2n-2}\theta_x}{\theta}\right)_x+\kappa\rho\left(\frac{r^{n-1}\theta_x}{\theta}\right)^2+\epsilon\rho\frac{(r^{n-1}u)_x^2}{\theta}\\&-\frac{2\mu(n-1)}{\theta}\left(r^{n-2}u^2\right)_x+\frac{k\lambda\phi Z}{\theta}.\end{aligned}\tag{2.3.127}$$

Then it follows that

$$\begin{aligned}&\mathcal{E}_t+\frac{\overline{\theta}}{\theta}\left(\epsilon\rho\left(r^{n-1}u\right)_x^2+\kappa\rho\frac{\left(r^{n-1}\theta_x\right)^2}{\theta}+k\lambda\phi Z\right)\\&\quad=\left[\epsilon\rho\left(r^{n-1}u\right)\left(r^{n-1}u\right)_x+d\lambda\rho^2r^{2n-2}Z_x+\kappa\left(1-\frac{\overline{\theta}}{\theta}\right)r^{2n-2}\rho\theta_x-\left(P-P(\overline{v},\overline{\theta})\right)\left(r^{n-1}u\right)\right]_x\\&\qquad-2\mu(n-1)\left(1-\frac{\overline{\theta}}{\theta}\right)\left(r^{n-2}u^2\right)_x,\end{aligned}\tag{2.3.128}$$

$$\begin{aligned}&\left[\frac{1}{2}\epsilon\left(\frac{\rho_x}{\rho}\right)^2+\frac{\rho_x}{\rho}\left(r^{1-n}u\right)\right]_t+(n-1)r^{-n}u^2\frac{\rho_x}{\rho}+\frac{P_\rho\rho_x^2}{\rho}\\&\quad=\rho\left(r^{1-n}u\right)_x\left(r^{n-1}u\right)_x-P_\theta\frac{\rho_x\theta_x}{\rho}-\left(\rho\left(r^{1-n}u\right)\left(r^{n-1}u\right)_x\right)_x.\end{aligned}\tag{2.3.129}$$

We denote

$$M(x,t)=e^{\gamma t}\left(\mathcal{E}+\frac{\alpha\epsilon}{2}\left(\frac{\rho_x}{\rho}\right)^2+\frac{\alpha\rho_x}{\rho}\left(r^{1-n}u\right)\right)$$

with $\gamma,\, \alpha > 0$ being positive constants to be specified later. Then, it follows directly from (2.3.128) and (2.3.129) that

$$
\begin{aligned}
&\frac{\partial}{\partial t} M(x,t) + e^{\gamma t}\frac{\overline{\theta}}{\theta}\left(\epsilon\rho\left(r^{n-1}u\right)_x^2 + \kappa\rho\frac{\left(r^{n-1}\theta_x\right)^2}{\theta} + k\lambda\phi Z\right)\\
&\quad + \alpha e^{\gamma t}\left((n-1)r^{-n}u^2\frac{\rho_x}{\rho} + P_\rho\frac{\rho_x^2}{\rho} + P_\theta\frac{\rho_x\theta_x}{\rho} - \rho\left(r^{1-n}u\right)_x\left(r^{n-1}u\right)_x\right)\\
&= \gamma e^{\gamma t}\left(\mathcal{E} + \frac{1}{2}\alpha\mu\left(\frac{\rho_x}{\rho}\right)^2 + \alpha\left(r^{1-n}u\right)\frac{\rho_x}{\rho}\right)\\
&\quad + e^{\gamma t}\Big(\rho\left(\epsilon\left(r^{n-1}u\right) - \alpha\left(r^{1-n}u\right)\right)\left(r^{n-1}u\right)_x + d\lambda\rho^2 r^{2n-2}Z_x\\
&\quad + \kappa\left(1-\frac{\overline{\theta}}{\theta}\right)r^{2n-2}\rho\theta_x - \left(P - P(\overline{v},\overline{\theta})\right)\left(r^{n-1}u\right)\Big)_x\\
&\quad - 2\mu(n-1)\left(1-\frac{\overline{\theta}}{\theta}\right)\left(r^{n-2}u^2\right)_x e^{\gamma t}.
\end{aligned}
\tag{2.3.130}
$$

Integrating (2.3.130) over Q_t and making use of lemma 2.3.13, together with the Young inequality and the Poincaré inequality, we can obtain that for any $\alpha > 0$,

$$
\begin{aligned}
M_1(t) &\equiv \int_0^L M(x,t)\,dx + \int_0^t\int_0^L e^{\gamma s}\frac{\overline{\theta}}{\theta}\left(\epsilon\rho\left(r^{n-1}u\right)_x^2 + \kappa\rho\frac{\left(r^{n-1}\theta_x\right)^2}{\theta} + k\lambda\phi Z\right)dxds\\
&\quad + \alpha\int_0^t\int_0^L e^{\gamma s}\left((n-1)r^{-n}u^2\frac{\rho_x}{\rho} + P_\rho\frac{\rho_x^2}{\rho} - \rho\left(r^{1-n}u\right)_x\left(r^{n-1}u\right)_x + P_\theta\frac{\rho_x\theta_x}{\rho}\right)dxds\\
&= \int_0^L M(x,0)\,dx + \gamma\int_0^t\int_0^L e^{\gamma s}\left(\mathcal{E} + \frac{\alpha\epsilon}{2}\left(\frac{\rho_x}{\rho}\right)^2 + \frac{\alpha\rho_x}{\rho}\left(r^{1-n}u\right)\right)dxds\\
&\quad - 2\mu(n-1)\int_0^t\int_0^L e^{\gamma s}\left(1-\frac{\overline{\theta}}{\theta}\right)\left(r^{n-2}u^2\right)_x dxds\\
&\le C_1 + \gamma\int_0^t\int_0^L e^{\gamma s}\left(\mathcal{E} + \frac{\alpha\epsilon}{2}\left(\frac{\rho_x}{\rho}\right)^2 + \frac{\alpha\rho_x}{\rho}\left(r^{1-n}u\right)\right)dxds + C_1\alpha\int_0^t e^{\gamma s}\|u_x(s)\|^2 ds\\
&\quad + C_1\alpha^{-1}\int_0^t e^{\gamma s}\|u(s)\|_{L^\infty}^2\|\theta_x(s)\|^2 ds\\
&\le C_1 + C_1\gamma\int_0^t e^{\gamma s}\left(\|u\|^2 + \|Z\|_{L^1} + \|v-\overline{v}\|^2 + \|\eta-\overline{\eta}\|^2 + \|\rho_x\|^2\right)(s)\,ds\\
&\quad + C_1\alpha\int_0^t e^{\gamma s}\|u_x(s)\|^2 ds + C_1\alpha^{-1}\int_0^t e^{\gamma s}\|u(s)\|_{L^\infty}^2\|\theta_x(s)\|^2 ds.
\end{aligned}
\tag{2.3.131}
$$

On the other hand, we know the following facts

$$
\left\|\left(r^{1-n}u\right)_x\right\| \le C_1\left\|\left(r^{n-1}u\right)_x\right\|, \quad \|u\|_{L^4}^4 \le C_1\left\|\left(r^{n-1}u\right)_x\right\|^2
$$

and

$$
P_\theta\frac{\rho_x\theta_x}{\rho} \ge -P_\rho\frac{\rho_x^2}{2\rho} - \frac{\rho\left(r^{n-1}\theta_x\right)^2}{2r^{2n-2}\theta}, \quad P_\rho = R\theta > 0.
$$

Thus we can conclude from lemma 2.3.13 that

$$\begin{aligned}
M_1(t) \geq{}& C_1^{-1} e^{\gamma t}\left(\frac{1}{2}\|u\|^2+\|Z\|_{L^1}+\|v-\overline{v}\|^2+\|\eta-\overline{\eta}\|^2\right)\\
&+\alpha e^{\gamma t}\int_0^L\left(\frac{\epsilon}{2}\left(\frac{\rho_x}{\rho}\right)^2+\left(r^{1-n}u\right)\frac{\rho_x}{\rho}\right)dx\\
&+C_1^{-1}\int_0^t e^{\gamma s}\int_0^L\left(C_1^{-1}\left(\epsilon\frac{\overline{\theta}}{\theta}-C_1\alpha\right)\rho\left(r^{n-1}u\right)_x^2\right.\\
&\left.+\left(\kappa\frac{\overline{\theta}}{\theta}-\frac{C_1\alpha}{2a^{2n-2}}\right)\rho\frac{\left(r^{n-1}\theta_x\right)^2}{\theta}+\frac{\alpha R\theta\rho_x^2}{2\rho}\right)dxds\\
&+C_1^{-1}\int_0^t\int_0^L e^{\gamma s}\phi Z dxds\\
\geq{}& C_1^{-1}e^{\gamma t}\left(\left(\frac{1}{2}-\frac{\alpha a^{2-2n}}{\epsilon}\right)\|u\|^2+\|Z\|_{L^1}+\|v-\overline{v}\|^2+\|\eta-\overline{\eta}\|^2+\|\rho_x\|^2\right)\\
&+C_1^{-1}\int_0^t e^{\gamma s}\int_0^L\left(C_1^{-1}\left(\epsilon\frac{\overline{\theta}}{\theta}-C_1\alpha\right)\rho\left(r^{n-1}u\right)_x^2\right.\\
&\left.+\left(\kappa\frac{\overline{\theta}}{\theta}-\frac{C_1\alpha}{2a^{2n-2}}\right)\rho\frac{\left(r^{n-1}\theta_x\right)^2}{\theta}+\frac{\alpha R\theta\rho_x^2}{2\rho}\right)dxds\\
&+C_1^{-1}\int_0^t\int_0^L e^{\gamma s}\phi Z dxds \qquad (2.3.132)
\end{aligned}$$

which, taking $\alpha > 0$ small enough so that

$$\frac{1}{2}-\frac{\alpha a^{2-2n}}{\epsilon}>0,\quad \epsilon\frac{\overline{\theta}}{\theta}-C_1\alpha>0,\quad \kappa\frac{\overline{\theta}}{\theta}-\frac{C_1\alpha}{2a^{2n-2}}>0$$

implies

$$\begin{aligned}
M_1(t) \geq{}& C_1^{-1}e^{\gamma t}\left(\|u(t)\|^2+\|Z(t)\|_{L^1}+\|v(t)-\overline{v}\|^2+\|\eta(t)-\overline{\eta}\|^2+\|\rho_x(t)\|^2+\|v_x(t)\|^2\right)\\
&+C_1^{-1}\int_0^t e^{\gamma s}\left(\|u_x\|^2+\|\theta_x\|^2+\|\rho_x\|^2+\|v_x\|^2\right)(s)ds\\
&+C_1^{-1}\int_0^t\int_0^L e^{\gamma s}\phi Z dxds. \qquad (2.3.133)
\end{aligned}$$

By the mean value theorem and the Poincaré inequality, it follows from lemma 2.3.1, (2.3.3) that

$$\begin{aligned}\|\theta-\overline{\theta}\| &\le C_1\left(\|v-\overline{v}\|+\left\|e(v,\theta)-e(\overline{v},\overline{\theta})\right\|\right)\\ &\le C_1\left(\|v-\overline{v}\|+\|e-\overline{e}\|+\|u\|+\|Z\|_{L^1}\right)\\ &\le C_1(\|v_x\|+\|e_x\|+\|u_x\|+\|Z\|)\\ &\le C_1(\|v_x\|+\|\theta_x\|+\|u_x\|+\|Z\|) \end{aligned}\tag{2.3.134}$$

and

$$\begin{aligned}\|\eta-\overline{\eta}\| &= \left\|\eta(v,\theta)-\eta\left(\overline{v},\overline{\theta}\right)\right\|\\ &\le C_1\left(\|v-\overline{v}\|+\left\|\theta-\overline{\theta}\right\|\right)\\ &\le C_1(\|v_x\|+\|\theta_x\|+\|u_x\|+\|Z\|)\end{aligned}\tag{2.3.135}$$

with $\overline{e}=\frac{1}{L}\int_0^L e(v,\theta)\,dx$.

It follows from (2.3.131)–(2.3.135) that

$$\begin{aligned}&e^{\gamma t}\Big(\|u(t)\|^2+\|Z(t)\|_{L^1}+\|Z(t)\|^2+\|v(t)-\overline{v}\|^2+\|\eta(t)-\overline{\eta}\|^2+\|\rho_x(t)\|^2+\|v_x(t)\|^2\Big)\\ &\quad+\int_0^t e^{\gamma s}\Big(\|u_x\|^2+\|\theta_x\|^2+\|\rho_x\|^2+\|v_x\|^2\Big)(s)\,ds+C_1^{-1}\int_0^t\int_0^L e^{\gamma s}\phi Z dxds\\ &\quad\le C_1+C_1\gamma\int_0^t e^{\gamma s}\Big(\|u_x\|^2+\|\theta_x\|^2+\|\rho_x\|^2+\|v_x\|^2+\|Z\|_{L^1}+\|Z\|^2\Big)(s)\,ds\\ &\qquad+C_1\alpha^{-1}\int_0^t e^{\gamma s}\|u(s)\|_{L^\infty}^2\|\theta_x(s)\|^2\,ds.\end{aligned}\tag{2.3.136}$$

Multiplying (2.2.30) by Z and integrating the resultant over $[0, L]$ with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\|Z\|^2+\int_0^L\frac{dr^{2n-2}}{v^2}Z_x^2\,dx+k\int_0^L\phi Z^2\,dx=0$$

which implies for some constant $\gamma_0>0$

$$\frac{d}{dt}\|Z(t)\|^2+\gamma_0\|Z_x(t)\|^2+\gamma_0\|Z(t)\|^2\le 0,$$

such that we can get for any $t>0$,

$$e^{\gamma_0 t}\|Z(t)\|^2+\int_0^t e^{\gamma_0 s}\Big(\|Z_x(s)\|^2+\|Z(s)\|^2\Big)\,ds\le C_1.\tag{2.3.137}$$

Thus we derive from (2.3.136) and (2.3.137) that there exists a constant $0<\gamma_1'=\gamma_1'(C_1)<\frac{\gamma_0}{2}$ such that for all $\gamma\in(0,\,\gamma_1']$ and for any $t>0$,

$$
\begin{aligned}
& e^{\gamma t}\Big(\|u\|^2+\|Z\|_{L^1}+\|Z\|^2+\|v-\overline{v}\|^2+\|\eta-\overline{\eta}\|^2+\big\|\theta-\overline{\theta}\big\|^2+\|\rho-\overline{\rho}\|^2+\|v_x\|^2\Big)\\
&\quad +\int_0^t e^{\gamma s}\Big(\|u_x\|^2+\|\theta_x\|^2+\|\rho_x\|^2+\|v_x\|^2+\|Z\|^2+\|Z_x\|^2+\int_0^L \phi Z dx\Big)ds\\
&\qquad \le C_1+C_1\int_0^t e^{\gamma s}\|u\|_{L^\infty}^2\|\theta_x\|^2 ds
\end{aligned}
$$

which gives (2.3.125). □

Lemma 2.3.15. *Under assumptions in theorem 2.2.1, there exists a positive constant* $\gamma\in(0,\,\gamma_1']$*, such that for any* $t>0$*,*

$$
\begin{aligned}
& e^{\gamma t}\Big(\|u_x(t)\|^2+\|\theta_x(t)\|^2+\|Z_x(t)\|^2\Big)\\
&\quad +\int_0^t e^{\gamma s}\Big(\|u_{xx}\|^2+\|\theta_{xx}\|^2+\|Z_{xx}\|^2+\|u_t\|^2+\|\theta_t\|^2+\|Z_t\|^2\Big)(s)\,ds\le C_1. \qquad (2.3.138)
\end{aligned}
$$

Proof. First, we easily verify

$$
\epsilon\rho u_{xx}\big(r^{n-1}u\big)_{xx}\ge C_1^{-1}u_{xx}^2-C_1\big(u^2+u_x^2+v_x^2\big). \qquad (2.3.139)
$$

Multiplying (2.2.28) by $-e^{\gamma t}u_{xx}$, integrating the result over $[0,\,L]$ and using the Gagliardo-Nirenberg inequality, the Young inequality and (2.3.139), we can derive that

$$
\begin{aligned}
\frac{d}{dt}\Big(e^{\gamma t}\|u_x\|^2\Big)+C_1^{-1}e^{\gamma t}\int_0^L u_{xx}^2dx &\le \frac{\gamma}{2}e^{\gamma t}\|u_x\|^2+C_1e^{\gamma t}\Big(\|u_x\|^2+\|v_x\|^2\Big)\\
&\quad +C_1e^{\gamma t}\big(\|u_x\|+\|u_x\|_{L^\infty}\|v_x\|+\|v_x\|+\|\theta_x\|\big)\|u_{xx}\|\\
&\le C_1\gamma e^{\gamma t}\|u_{xx}\|^2+C_1e^{\gamma t}\Big(\|u_x\|^2+\|v_x\|^2+\|\theta_x\|^2\Big)
\end{aligned}
$$

which implies

$$
\frac{d}{dt}\Big(e^{\gamma t}\|u_x\|^2\Big)+\frac{1}{2C_1}e^{\gamma t}\|u_{xx}\|^2\le C_1\gamma^{-1}e^{\gamma t}\Big(\|u_x\|^2+\|v_x\|^2+\|\theta_x\|^2\Big). \qquad (2.3.140)
$$

We can easily derive from (2.2.29) that

$$
\begin{aligned}
\theta_t=&-\frac{e_v(r^{n-1}u)_x}{e_\theta}+\frac{(r^{2n-2}\kappa\theta_x)_x}{e_\theta v}-\frac{r^{2n-2}\kappa\theta_x v_x}{e_\theta v^2}+\epsilon\frac{(r^{n-1}u)_x^2}{e_\theta v}\\
&-2\mu(n-1)\frac{(r^{n-2}u^2)_x}{e_\theta}-\frac{P(r^{n-1}u)_x}{e_\theta}+\frac{k\lambda\phi Z}{e_\theta}. \qquad (2.3.141)
\end{aligned}
$$

Noting that the fact

$$
\rho\kappa\big(r^{2n-2}\theta_x\big)_x\theta_{xx}\ge C_1^{-1}\theta_{xx}^2-C_1\theta_x^2 \qquad (2.3.142)
$$

and multiplying (2.3.141) by $-e^{\gamma t}\theta_{xx}$, then integrating the result over $[0,\, L]$, using the Young inequality and the Sobolev inequality, we derive that

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{\gamma t}\|\theta_x\|^2\Big) + C_1^{-1}e^{\gamma t}\int_0^L \theta_{xx}^2\,dx \\
&\quad\le \frac{\gamma}{2}e^{\gamma t}\|\theta_x\|^2 + C_1 e^{\gamma t}\|\theta_x\|^2 + C_1 e^{\gamma t}(\|u\| + \|u_x\| + \|\theta_x\|_{L^\infty}\|v_x\| \\
&\qquad + \|\theta_x\|_{L^\infty}\|\theta_x\| + \|u_x\|_{L^\infty} + \|Z\|_{L^1})\|\theta_{xx}\| \\
&\quad\le \frac{\gamma}{2}e^{\gamma t}\|\theta_x\|^2 + C_1 e^{\gamma t}\|\theta_x\|^2 + C_1 e^{\gamma t}(\|\theta_x\|^{\frac{1}{2}}\|\theta_{xx}\|^{\frac{3}{2}}\|v_x\| + \|\theta_x\|^{\frac{3}{2}}\|\theta_{xx}\|^{\frac{3}{2}} + \|\theta_{xx}\|\|u_x\| \\
&\qquad + \|u_x\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}}\|\theta_{xx}\| + \|Z\|\|\theta_{xx}\|) \\
&\quad\le \frac{\gamma}{2}e^{\gamma t}\|\theta_{xx}\|^2 + \gamma e^{\gamma t}\|u_{xx}\|^2 + C_1(\gamma)e^{\gamma t}\Big(\|\theta_x\|^2 + \|u_x\|^2 + \|Z\|^2\Big)
\end{aligned}$$

which, taking $\gamma > 0$ small enough, gives

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{\gamma t}\|\theta_x\|^2\Big) + \frac{1}{2C_1}e^{\gamma t}\|\theta_{xx}\|^2 \\
&\quad\le \gamma e^{\gamma t}\Big(\|\theta_{xx}\|^2 + \|u_{xx}\|^2\Big) + C_1(\gamma)e^{\gamma t}\Big(\|u_x\|^2 + \|Z\|^2 + \|\theta_x\|^2\Big). \qquad (2.3.143)
\end{aligned}$$

Similarly, we derive from (2.2.30) by multiplying (2.2.30) by $-e^{\gamma t}Z_{xx}$ over $[0,\, L]$ that

$$\frac{d}{dt}\Big(e^{\gamma t}\|Z_x\|^2\Big) + \frac{1}{2C_1}e^{\gamma t}\|Z_{xx}\|^2 \le C_1\gamma^{-1}e^{\gamma t}\Big(\|Z_x\|^2 + \|Z\|^2 + \|v_x\|^2\Big). \qquad (2.3.144)$$

Adding (2.3.140), (2.3.143) and (2.3.144) up, we can get

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{\gamma t}\Big(\|u_x\|^2 + \|\theta_x\|^2 + \|Z_x\|^2\Big)\Big) + \frac{1}{2C_1}e^{\gamma t}\Big(\|u_{xx}\|^2 + \|\theta_{xx}\|^2 + \|Z_{xx}\|^2\Big) \\
&\quad\le \gamma e^{\gamma t}\Big(\|u_{xx}\|^2 + \|\theta_{xx}\|^2\Big) + C_1\gamma^{-1}e^{\gamma t}\Big(\|u_x\|^2 + \|v_x\|^2 + \|\theta_x\|^2 + \|Z_x\|^2 + \|Z\|^2\Big)
\end{aligned}$$

which, taking $0<\gamma\Big(<\frac{1}{2C_1}\Big)$ small enough and using lemma 2.3.14, yields for any $t > 0$,

$$\begin{aligned}
&e^{\gamma t}\Big(\|u_x\|^2 + \|\theta_x\|^2 + \|Z_x\|^2\Big) + \int_0^t e^{\gamma s}\Big(\|u_{xx}\|^2 + \|\theta_{xx}\|^2 + \|Z_{xx}\|^2\Big)(s)\,ds \\
&\quad\le C_1 + C_1\int_0^t e^{\gamma s}\|u(s)\|_{L^\infty}^2\|\theta_x(s)\|^2\,ds. \qquad (2.3.145)
\end{aligned}$$

Applying the Gronwall inequality to the above inequality, we easily obtain for any $t > 0$,

$$e^{\gamma t}\Big(\|u_x\|^2+\|\theta_x\|^2+\|Z_x\|^2\Big)+\int_0^t e^{\gamma s}\Big(\|u_{xx}\|^2+\|\theta_{xx}\|^2+\|Z_{xx}\|^2\Big)(s)\,ds\le C_1. \tag{2.3.146}$$

From (2.2.28)–(2.2.30) and the embedding theorem, we derive

$$\|u_t\|\le C_1(\|u_{xx}\|+\|\theta_x\|+\|v_x\|), \tag{2.3.147}$$

$$\|\theta_t\|\le C_1(\|\theta_{xx}\|+\|\theta_x\|+\|v_x\|+\|u_{xx}\|+\|u_x\|+\|Z\|) \tag{2.3.148}$$

and

$$\|Z_t\|\le C_1(\|Z_{xx}\|+\|Z_x\|+\|v_x\|+\|Z\|). \tag{2.3.149}$$

Then it follows from (2.3.146)–(2.3.149) and lemma 2.3.14 that for any $t>0$,

$$\int_0^t e^{\gamma s}\Big(\|u_t\|^2+\|\theta_t\|^2+\|Z_t\|^2\Big)(s)\,ds\le C_1 \tag{2.3.150}$$

which, together with (2.3.146), implies (2.3.138). □

Till now we have completed the proof of theorem 2.2.1. □

2.4 Global Existence and Exponential Stability in $\mathcal{H}^2$

In this section, we shall show the global existence and exponential stability of solutions to the problem (2.2.27)–(2.2.32) on $\mathcal{H}^2$ under some relative assumptions and complete the proof of theorem 2.2.2, which can be shown by the following lemmas.

2.4.1 Global Existence of Solutions in $\mathcal{H}^2$

In this subsection, we complete the proof of global existence on $\mathcal{H}^2$ by a series of lemmas. First, we estimate the second-order derivatives of u and θ.

Lemma 2.4.1. *Under assumptions of theorem 2.2.2, there holds that for any $t>0$,*

$$\|u_t(t)\|^2+\|u_{xx}(t)\|^2+\|u_x(t)\|_{L^\infty}^2+\int_0^t\|u_{xt}(s)\|^2ds\le C_2, \tag{2.4.1}$$

$$\|\theta_t(t)\|^2+\|\theta_{xx}(t)\|^2+\|\theta_x(t)\|_{L^\infty}^2+\int_0^t\|\theta_{xt}(s)\|^2ds\le C_2. \tag{2.4.2}$$

Proof. Differentiating (2.2.28) with respect to t,

$$u_{tt} = (n-1)r^{n-2}r_t\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_x + r^{n-1}\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_{xt}. \tag{2.4.3}$$

Multiplying (2.4.3) by u_t and integrating the resultant over Q_t, using integrating by parts, the Cauchy inequality and theorem 2.2.1, we can arrive at

$$\int_0^t\int_0^L u_{tt}u_t dxds \geq \frac{1}{2}\int_0^L u_t^2 dx - C_2, \tag{2.4.4}$$

$$\begin{aligned}
&\int_0^t\int_0^L u_t r^{n-1}\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_{xt} dxds\\
&\quad = \int_0^t\int_0^L u_t r^{n-1}\left(\epsilon\frac{(r^{n-1}u)_{xt}}{v} - \epsilon\frac{(r^{n-1}u)_x v_t}{v^2} - R\frac{\theta_t}{v} + R\frac{\theta v_t}{v^2} + 4a\theta^3\theta_t\right)_x dxds\\
&\quad \leq -C_2^{-1}\int_0^t\int_0^L u_{xt}^2 dxds + C_2\varepsilon\int_0^t\int_0^L u_{xt}^2 dxds\\
&\qquad + C_2(\varepsilon)\int_0^t\int_0^L \left(u_x^2 + u_t^2 + \theta_t^2 + u_x^4\right) dxds\\
&\quad \leq C_2 - C_2^{-1}\int_0^t\int_0^L u_{xt}^2 dxds + C_2(\varepsilon)\int_0^t \|u_x\|_{L^\infty}^2\int_0^L u_x^2 dxds\\
&\quad \leq C_2 - C_2^{-1}\int_0^t\int_0^L u_{xt}^2 dxds
\end{aligned} \tag{2.4.5}$$

and

$$\begin{aligned}
&\int_0^t\int_0^L (n-1)uu_t r^{n-2}\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_x dxds\\
&\quad \leq C_2\int_0^t\int_0^L \left(u_t^2 + u_x^2v_x^2 + u_{xx}^2 + \theta_x^2 + \theta_x^2v_x^2\right) dxds\\
&\quad \leq C_2 + C_2\int_0^t \|u_x\|_{L^\infty}^2\int_0^L v_x^2 dxds + C_2\int_0^t \|\theta_x\|_{L^\infty}^2\int_0^L v_x^2 dxds\\
&\quad \leq C_2 + C_2\int_0^t\left(\|u_x\|^2 + \|u_{xx}\|^2\right)ds + C_2\int_0^t\left(\|\theta_x\|^2 + \|\theta_{xx}\|^2\right)ds\\
&\quad \leq C_2.
\end{aligned} \tag{2.4.6}$$

Combining (2.4.3)–(2.4.6), we can obtain

$$\|u_t(t)\|^2 + \int_0^t \|u_{xt}(s)\|^2 ds \leq C_2 \tag{2.4.7}$$

which, by noting the fact that from (2.2.28),

$$\|u_{xx}(t)\| \le C_2\big(\|u_t\| + \|v_x\| + \|u_x\|_{H^1} + \|\theta\|_{H^1}\big), \tag{2.4.8}$$

gives

$$\|u_{xx}(t)\| \le C_2. \tag{2.4.9}$$

Hence (2.4.1) follows from (2.4.7) using the embedding theorem.
Similarly, differentiating (2.2.29) with respect to t, we can get

$$\begin{aligned} c_v\theta_{tt} + \big(av_t\theta^4 + 4av\theta^3\theta_t\big)_t = {}& \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_{xt} + \epsilon\left(\frac{(r^{n-1}u)_x^2}{v}\right)_t \\ & - 2\mu(n-1)\big(r^{n-2}u^2\big)_{xt} - \Big(P\big(r^{n-1}u\big)_x\Big)_t + k\lambda(\phi Z)_t. \end{aligned} \tag{2.4.10}$$

Multiplying (2.4.10) by θ_t and integrating the result over Q_t, we have

$$\int_0^t\int_0^L c_v\theta_{tt}\theta_t dxds \ge \frac{c_v}{2}\int_0^L \theta_t^2 dx - C_2, \tag{2.4.11}$$

$$\begin{aligned} &\int_0^t\int_0^L \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_{xt}\theta_t dxds \\ &\quad\le -C_2^{-1}\int_0^t\int_0^L \theta_{xt}^2 dxds + C_2\varepsilon\int_0^t\int_0^L \theta_{xt}^2 dxds + C_2(\varepsilon)\int_0^t\int_0^L \theta_x^2 v_t^2 dxds \\ &\quad\le -C_2^{-1}\int_0^t\int_0^L \theta_{xt}^2 dxds + C_2\int_0^t \|u_x\|_{L^\infty}^2\int_0^L \theta_x^2 dxds \\ &\quad\le C_2 - C_2^{-1}\int_0^t\int_0^L \theta_{xt}^2 dxds, \end{aligned} \tag{2.4.12}$$

$$\begin{aligned} &\int_0^t\int_0^L \left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_t \big(r^{n-1}u\big)_x\theta_t dxds \\ &\quad\le C_2\int_0^t\int_0^L \big(u_{xt}^2 + u_x^4 + \theta_t^2 + \theta^2 u_x^2 + u_x^2\theta_t^2\big)dxds \\ &\quad\le C_2 + C_2\int_0^t \|u_x\|_{L^\infty}^2\int_0^L \|u_x\|^2 dxds + C_2\int_0^t \|u_x\|_{L^\infty}^2\int_0^L \|\theta_t\|^2 dxds \\ &\quad\le C_2 + C_2\int_0^t \|u_x\|_{L^\infty}^2\int_0^L \|\theta_t\|^2 dxds \end{aligned} \tag{2.4.13}$$

and

$$\begin{aligned}
&\int_0^t\int_0^L (-2\mu(n-1))\big(r^{n-2}u^2\big)_{xt}\theta_t dxds\\
&\quad\le C_2\int_0^t\int_0^L \big(\theta_t^2+u_x^2u_t^2+u^2u_{xt}^2\big)dxds\\
&\quad\le C_2+C_2\int_0^t\|u_x\|_{L^\infty}^2\int_0^L\|u_x\|^2dxds+C_2\int_0^t\int_0^L u_{xt}^2dxds\\
&\quad\le C_2.
\end{aligned}\tag{2.4.14}$$

From (2.2.33), (2.3.1) and theorem 2.2.1, it follows

$$\left\|\epsilon\frac{(r^{n-1}u)_x}{v}-P\right\|_{L^\infty}\le C_2,\tag{2.4.15}$$

$$\left\|\big(r^{n-1}u\big)_{xt}\right\|\le C_2(\|u_x\|+\|u_{xt}\|),\tag{2.4.16}$$

$$\left|\int_0^L\theta_t dx\right|\le\left|\int_0^L\left(\epsilon\frac{(r^{n-1}u)_x}{v}-P\right)\big(r^{n-1}u\big)_x dx\right|\le C_2\|u_x\|\tag{2.4.17}$$

which, by the Poincaré inequality, gives

$$\|\theta_t\|\le\left|\int_0^L\theta_t dx\right|+C_1\|\theta_{xt}\|\le C_2(\|u_x\|+\|\theta_{xt}\|).\tag{2.4.18}$$

Thus from (2.4.15)–(2.4.18), we derive for $\varepsilon>0$ small enough,

$$\begin{aligned}
\int_0^L\left(\epsilon\frac{(r^{n-1}u)_x}{v}-P\right)\big(r^{n-1}u\big)_{tx}\theta_t dx&\le C_2\left\|\big(r^{n-1}u\big)_{tx}\right\|\|\theta_t\|\\
&\le C_2(\|u_x\|+\|u_{xt}\|)(\|u_x\|+\|\theta_{xt}\|)\\
&\le\varepsilon\|\theta_{xt}\|^2+C_2(\|u_x\|+\|u_{xt}\|),
\end{aligned}\tag{2.4.19}$$

$$\int_0^t\int_0^L k\lambda(\phi Z)_t\theta_t dxds\le C_2\int_0^t\int_0^L\big(\theta_t^2+Z_t^2\big)dxds\le C_2,\tag{2.4.20}$$

$$\begin{aligned}
\int_0^t\int_0^L\Big(\big(av_t\theta^4\big)_t+\big(4av\theta^3\big)_t\theta_t\Big)\theta_t dxds&\le C_2\int_0^t\int_0^L\Big(|v_t|\theta_t^2+\theta_t^2+|\theta_t|^3\Big)dxds\\
&\le\varepsilon\int_0^t\|\theta_{xt}\|^2ds+C_2,
\end{aligned}\tag{2.4.21}$$

$$\int_0^t\int_0^L 4av\theta^3\theta_{tt}\theta_t dxds = \int_0^L (2av\theta^3\theta_t^2)\Big|_0^t dx - \int_0^t\int_0^L \left(2av\theta^3\right)_t \theta_t^2 dxds, \tag{2.4.22}$$

$$\begin{aligned}\int_0^t\int_0^L \left(2av\theta^3\right)_t\theta_t^2 dxds &\le C_2\int_0^t\int_0^L |v_t|\theta_t^2 dxds + C_2\int_0^t\int_0^L |\theta_t|^3 dxds\\ &\le C_2\int_0^t \|\theta_t\|_{L^\infty}^2 ds \le \varepsilon\int_0^t \|\theta_{xt}\|^2 ds + C_2.\end{aligned} \tag{2.4.23}$$

It thus follows from (2.4.22) and (2.4.23) that

$$\int_0^t\int_0^L 4av\theta^3\theta_{tt}\theta_t dxds \ge \frac{C_2}{2}\int_0^L \theta_t^2 dx - C_2 - \varepsilon\int_0^t \|\theta_{xt}\|^2 ds \tag{2.4.24}$$

which, together with (2.4.10)–(2.4.21), gives

$$\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(s)\|^2 ds \le C_2 + C_2\int_0^t \|u_x(s)\|_{L^\infty}^2 \int_0^L \|\theta_t(s)\|^2 dxds. \tag{2.4.25}$$

By the Gronwall inequality, we obtain

$$\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(s)\|^2 ds \le C_2 \tag{2.4.26}$$

which, by noting that from (2.2.29),

$$\|\theta_{xx}(t)\| \le C_2\left(\|\theta_t(t)\| + \|\theta(t)\|_{H^1} + \|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|Z(t)\|_{L^1}\right), \tag{2.4.27}$$

and the embedding theorem, implies (2.4.2). □

The next lemma concern the estimate on the second-order derivative of v.

Lemma 2.4.2. *Under assumptions of theorem 2.2.2, there holds that for any* $t > 0$,

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \le C_2. \tag{2.4.28}$$

Proof. Differentiating (2.2.28) with respect to x, we have

$$\begin{aligned}\epsilon\frac{d}{dt}\left(\frac{v_{xx}}{v}\right) + \frac{R\theta v_{xx}}{v^2} &= r^{1-n}u_{xt} + (1-n)r^{-n}vu_t + 2\epsilon\frac{v_x(r^{n-1}u)_{xx}}{v^2} - 2\epsilon\frac{v_x^2(r^{n-1}u)_x}{v^3}\\ &\quad - R\frac{\theta_{xx}}{v} + 2R\frac{v_x\theta_x}{v^2} - 2\frac{\theta v_x^2}{v^3} + 4a\theta^2\theta_x^2 + \frac{4}{3}a\theta^3\theta_{xx} := \mathcal{M}\end{aligned} \tag{2.4.29}$$

where

$$\|\mathcal{M}\| \le C_2\left(\|u_{xt}\| + \|u_t\| + \|v_x\| + \|u_x\|_{H^1} + \|\theta\|_{H^2}\right).$$

By theorem 2.2.1 and (2.4.1), we get for any $t > 0$,

$$\int_0^t \|\mathcal{M}(s)\|^2 ds \le C_2. \tag{2.4.30}$$

Multiplying (2.4.29) by $\frac{v_{xx}}{v}$, then integrating the result over Q_t, and using the Young inequality and (2.4.30), we can obtain (2.4.28). □

Now we establish the estimate on the second-order derivative of Z.

Lemma 2.4.3. *Under assumptions of theorem 2.2.2, the following estimate holds for any $t > 0$,*

$$\|Z_{xx}(t)\|^2 + \|Z(t)\|^2 + \|Z_x(t)\|^2_{L^\infty} + \int_0^t \|Z_{xt}(s)\|^2 ds \le C_2. \tag{2.4.31}$$

Proof. Differentiating (2.2.30) with respect to t, we have

$$Z_{tt} = d\left(\frac{r^{2n-2}Z_x}{v^2}\right)_{xt} - k\phi_t Z - k\phi Z_t. \tag{2.4.32}$$

Multiplying (2.4.32) by Z_t and then integrating the result over Q_t, we have

$$\int_0^t \int_0^L Z_{tt} Z_t \, dx ds = \frac{1}{2}\int_0^L Z_t^2 dx - C_2, \tag{2.4.33}$$

$$\begin{aligned}\int_0^t \int_0^L d\left(\frac{r^{2n-2}Z_x}{v^2}\right)_{xt} Z_t \, dx ds \le & - C_2^{-1}\int_0^t \int_0^L Z_{xt}^2 dx ds + C_2\varepsilon \int_0^t \int_0^L Z_{xt}^2 dx ds \\ & + C_2 \int_0^t \int_0^L Z_x^2 v_t^2 dx ds \\ \le & \, C_2 - C_2^{-1}\int_0^t \int_0^L Z_{xt}^2 dx ds, \end{aligned} \tag{2.4.34}$$

$$\int_0^t \int_0^L (k\phi_t Z + k\phi Z_t) Z_t \, dx ds \le C_2 \int_0^t \int_0^L \left(\theta_t^2 + Z_t^2\right) dx ds \le C_2. \tag{2.4.35}$$

Combining (2.4.32)–(2.4.35), we can obtain

$$\int_0^L Z_t^2 dx + \int_0^t \int_0^L Z_{xt}^2 dx ds \le C_2 \tag{2.4.36}$$

which, by noting that from (2.2.30),

$$\|Z_{xx}\| \le C_2\left(\|Z_t\| + \|Z_x\| + \|Z\|_{H^1} + \|v_x\|\right) \tag{2.4.37}$$

and the embedding theorem, implies (2.4.31). □

Based on lemmas 2.4.1–2.4.3, we easily obtain the following estimate from equations (2.2.28)–(2.2.30).

Lemma 2.4.4. *Under assumptions of theorem 2.2.2, the following estimate holds for any* $t > 0$,

$$\int_0^t \left(\|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|Z_{xxx}\|^2\right)(s)\,ds \le C_2. \tag{2.4.38}$$

Proof. Differentiating (2.2.28) with respect to x, we can get

$$\|u_{xxx}\| \le C_2\left(\|u_{xt}\| + \|u_x\|_{H^1} + \|v_x\|_{H^1} + \|\theta\|_{H^2}\right). \tag{2.4.39}$$

By theorem 2.2.1 and lemmas 2.4.1 and 2.4.2, we have

$$\int_0^t \|u_{xxx}(s)\|^2\,ds \le C_2. \tag{2.4.40}$$

Similarly, differentiating (2.2.29) with respect to x, we can obtain

$$\|\theta_{xxx}\| \le C_2\left(\|\theta_{xt}\| + \|\theta_x\|_{H^1} + \|v_x\|_{H^1} + \|u_x\|_{H^1} + \|Z_x\|\right). \tag{2.4.41}$$

By theorem 2.2.1 and lemmas 2.4.1 and 2.4.2, we have

$$\int_0^t \|\theta_{xxx}(s)\|^2\,ds \le C_2. \tag{2.4.42}$$

Also, differentiating (2.2.30) with respect to x, we can derive

$$\|Z_{xxx}\| \le C_2\left(\|Z_{xt}\| + \|v_x\|_{H^1} + \|Z_x\|_{H^1} + \|\theta_x\| + \|Z_x\|\right). \tag{2.4.43}$$

By theorem 2.2.1 and lemmas 2.4.2 and 2.4.3, we also obtain

$$\int_0^t \|Z_{xxx}(s)\|^2\,ds \le C_2 \tag{2.4.44}$$

which, together with (2.4.40) and (2.4.42), implies (2.4.38). □

The next lemma means the global existence of solutions in $\mathcal{H}^2$.

Lemma 2.4.5. *Under assumptions in theorem 2.2.2, the following estimate holds for any* $t > 0$,

$$\begin{aligned}
&\|r(t) - \overline{r}\|_{H^3}^2 + \|r_t\|_{H^2}^2 + \|v(t) - \overline{v}\|_{H^2}^2 + \|u(t) - \overline{u}\|_{H^2}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^2}^2 \\
&\quad + \|Z(t)\|_{H^2}^2 + \|v(t) - \overline{v}\|_{W^{1,\infty}}^2 + \|u(t) - \overline{u}\|_{W^{1,\infty}}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{W^{1,\infty}}^2 \\
&\quad + \|v_t\|^2 + \|u_t\|^2 + \|\theta_t\|^2 + \|Z_t\|^2 + \int_0^t (\|v - \overline{v}\|_{H^2}^2 + \|v_t\|_{H^2}^2 + \|u - \overline{u}\|_{H^3}^2 \\
&\quad + \|u_t\|_{H^1}^2 + \left\|\theta - \overline{\theta}\right\|_{H^3}^2 + \|\theta_t\|_{H^1}^2 + \|Z_t\|_{H^1}^2 + \|Z\|_{H^3}^2 \\
&\quad + \|r - \overline{r}\|_{H^3}^2 + \|r_t\|_{H^3}^2)(s)\,ds \le C_2.
\end{aligned} \tag{2.4.45}$$

Proof. By equations (2.2.27)–(2.2.30), theorem 2.2.1, lemmas 2.4.1–2.4.4 and the embedding theorem, we have

$$\begin{aligned}
&\|v_{xt}(t)\| \le C_2\|u(t)\|_{H^2} \le C_2, \quad \|v_{xxt}(t)\| \le C_2\|u(t)\|_{H^3},\\
&\int_0^t \|v_{xxt}\|^2 ds \le C_2 \int_0^t \|u\|_{H^3}^2 ds \le C_2, \quad \|r(t)\|_{H^3} \le C_2\|u(t)\|_{H^2} \le C_2,\\
&\|r_t(t)\|_{H^2} \le C_2\|v(t)\|_{H^2} \le C_2, \quad \|v(t)\|_{W^{1,\infty}} \le C_2\|v(t)\|_{H^2} \le C_2,\\
&\|u(t)\|_{W^{1,\infty}} \le C_2\|u(t)\|_{H^2} \le C_2, \quad \|\theta(t)\|_{W^{1,\infty}} \le C_2\|\theta(t)\|_{H^2} \le C_2.
\end{aligned}$$

Combining lemmas 2.4.1–2.4.3 with the above estimates, we obtain (2.4.45). □

2.4.2 Exponential Stability of Solutions in $\mathcal{H}^2$

The exponential stability of generalized global solution $(u(t), v(t), \theta(t), Z(t))$ in $\mathcal{H}^2$ can be obtained by the following two lemmas.

Lemma 2.4.6. *There exists a positive constant $\gamma_2' = \gamma_2'(C_2) \leq \gamma_1$ such that for any fixed $\gamma \in (0, \gamma_2']$, the following estimate holds for any $t > 0$,*

$$\begin{aligned}
&e^{\gamma t}\Big(\|u_t(t)\|^2 + \|\theta_t(t)\|^2 + \|Z_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|Z_{xx}(t)\|^2\Big)\\
&\quad + \int_0^t e^{\gamma s}\Big(\|u_{xt}\|^2 + \|\theta_{xt}\|^2 + \|Z_{xt}\|^2\Big)(s)\,ds \le C_2.
\end{aligned} \tag{2.4.46}$$

Proof. Multiplying (2.4.3) by $u_t e^{\gamma t}$ and then integrating the resulting equation over Q_t, using the Young inequality and (2.4.4)–(2.4.6), we have

$$e^{\gamma t}\|u_t(t)\|^2 + \int_0^t e^{\gamma s}\|u_{xt}(s)\|^2 ds \le C_2 + \int_0^t e^{\gamma s}\Big(\|u_t\|^2 + \|\theta_t\|^2 + \|u_x\|_{H^1}^2\Big)(s)\,ds \tag{2.4.47}$$

which, along with lemmas 2.4.1–2.4.5 and lemmas 2.3.13–2.3.15 and noting the estimate (2.4.8), implies that there exists a constant $\gamma_2' = \gamma_2'(C_2) \leq \gamma_1$ such that for any fixed $\gamma \in (0, \gamma_2']$,

$$e^{\gamma t}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2\Big) + \int_0^t e^{\gamma s}\|u_{xt}(s)\|^2 ds \le C_2. \tag{2.4.48}$$

In the same manner, multiplying (2.4.10) by $\theta_t e^{\gamma t}$ and (2.4.32) by $\theta_t e^{\gamma t}$, respectively, integrating the resultants over Q_t, using lemmas 2.4.1–2.4.5 and theorem 2.2.1 and (2.4.27), (2.4.37), we derive that

$$e^{\gamma t}\Big(\|\theta_t(t)\|^2 + \|Z_t(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|Z_{xx}(t)\|^2\Big) + \int_0^t e^{\gamma s}\Big(\|\theta_{xt}\|^2 + \|Z_{xt}\|^2\Big)(s)\,ds \le C_2 \tag{2.4.49}$$

which, together with (2.4.48), yields (2.4.46). The proof is now complete. □

Lemma 2.4.7. *There exists a positive constant $\gamma_2 = \gamma_2(C_2) \leq \gamma_2'$ such that for any fixed $\gamma \in (0, \gamma_2]$, the following estimate holds for any $t > 0$,*

$$e^{\gamma t}\|v_{xx}(t)\|^2 + \int_0^t e^{\gamma s}\Big(\|v_{xx}\|^2 + \|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|Z_{xxx}\|^2\Big)(s)\,ds \leq C_2. \tag{2.4.50}$$

Proof. Multiplying (2.4.29) by $\frac{v_{xx}}{v}e^{\gamma t}$ and choosing γ so small that $\gamma \leq \gamma_2 = \min\Big\{\gamma_2', \frac{1}{4C_1}\Big\}$ and using lemmas 2.4.1–2.4.6 and theorem 2.2.1, we conclude that

$$\begin{aligned} e^{\gamma t}\|v_{xx}(t)\|^2 + \int_0^t e^{\gamma s}\|v_{xx}(s)\|^2 ds &\leq C_2 + C_1\int_0^t e^{\gamma s}\Big(\|v_x\|^2 + \|\theta_{xx}\|^2 + \|u_{xt}\|_{H^1}^2\Big)(s)\,ds \\ &\leq C_2. \end{aligned} \tag{2.4.51}$$

Differentiating (2.2.28), (2.2.29) and (2.2.30) with respect to x, respectively, using lemmas 2.4.1–2.4.6, we easily get

$$\int_0^t e^{\gamma s}\Big(\|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|Z_{xxx}\|^2\Big)(s)\,ds \leq C_2 \tag{2.4.52}$$

which, along with (2.4.51), yields (2.4.50). The proof is thus complete. □

Till now we have completed the proof of theorem 2.2.2. □

2.5 Global Existence and Exponential Stability in $\mathcal{H}^4$

In this section, we shall establish the global existence and exponential stability of solutions to the problem (2.2.27)–(2.2.32) under some relative assumptions and complete the proof of theorem 2.2.3 by means of the following lemmas.

2.5.1 Global Existence of Solutions in $\mathcal{H}^4$

In this subsection, we shall prove the global existence of solutions in $\mathcal{H}^4$. Firstly, we give some estimates on u_{tt}, θ_{tt} and Z_{tt}.

Lemma 2.5.1. *Under assumptions of theorem 2.2.3, there holds that for any $t > 0$ and for $\varepsilon > 0$ small enough,*

$$\|u_{xt}(x,0)\| + \|\theta_{xt}(x,0)\| + \|Z_{xt}(x,0)\| \leq C_4, \tag{2.5.1}$$

$$\begin{aligned} &\|u_{tt}(x,0)\| + \|\theta_{tt}(x,0)\| + \|Z_{tt}(x,0)\| + \|u_{txx}(x,0)\| \\ &\quad + \|\theta_{txx}(x,0)\| + \|Z_{txx}(x,0)\| \leq C_4, \end{aligned} \tag{2.5.2}$$

$$\|u_{tt}(t)\|^2+\int_0^t\|u_{ttx}(s)\|^2ds\le C_4+C_4\int_0^t\left(\|\theta_{txx}\|^2+\|u_{txx}\|^2\right)(s)\,ds, \tag{2.5.3}$$

$$\begin{aligned}\|\theta_{tt}(t)\|^2+\int_0^t\|\theta_{ttx}(s)\|^2ds\le{}& C_4+C_2\varepsilon^{-1}\int_0^t\|\theta_{txx}(s)\|^2ds\\&+C_1\varepsilon\int_0^t\left(\|u_{ttx}\|^2+\|u_{txx}\|^2+\|Z_{txx}\|^2\right)(s)\,ds,\end{aligned} \tag{2.5.4}$$

$$\|Z_{tt}(t)\|^2+\int_0^t\|Z_{ttx}(s)\|^2ds\le C_4+C_4\int_0^t\left(\|\theta_{txx}\|^2+\|Z_{txx}\|^2\right)(s)\,ds. \tag{2.5.5}$$

Proof. Differentiating (2.2.28), (2.2.29) and (2.2.30) with respect to x, respectively, using lemmas 2.3.10, 2.4.1 and 2.4.3, we can get

$$\|u_{xt}(t)\|\le C_2\left(\|u_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^1}+\|v_x(t)\|_{H^1}\right), \tag{2.5.6}$$

$$\|\theta_{xt}(t)\|\le C_2\left(\|\theta_x(t)\|_{H^2}+\|v_x(t)\|_{H^1}+\|u_x(t)\|_{H^1}+\|\theta_x(t)\|+\|Z_x(t)\|\right), \tag{2.5.7}$$

$$\begin{aligned}\|Z_{xt}(t)\|\le{}& C_2(\|Z_{xxx}(t)\|+\|Z_x(t)\|_{L^\infty}\|v_{xx}(t)\|+\|v_x(t)\|_{L^\infty}\|Z_{xx}(t)\|\\&+\|v_x(t)\|_{L^\infty}\|Z_x(t)\|+\|\theta_x(t)\|+\|Z_x(t)\|)\\\le{}& C_2\left(\|Z_x(t)\|_{H^2}+\|\theta_x(t)\|+\|v_x(t)\|_{H^1}\right).\end{aligned} \tag{2.5.8}$$

Combining (2.5.6)–(2.5.8), we easily obtain (2.5.1).

Similarly, differentiating (2.2.28), (2.2.29) and (2.2.30) with respect to x twice, respectively, using lemmas 2.3.10, 2.4.1 and 2.4.3, we can derive

$$\begin{aligned}\|u_{xxt}(t)\|\le{}& C_2(\|u_x(t)\|_{H^3}+\|v_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}+\|u_x(t)\|_{L^\infty}\|v_{xxx}(t)\|\\&+\|v_x(t)\|_{L^\infty}\|u_{xxx}(t)\|+\|u_{xx}(t)\|_{L^\infty}\|v_{xx}(t)\|)\\\le{}& C_2\left(\|u_x(t)\|_{H^3}+\|v_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}\right),\end{aligned} \tag{2.5.9}$$

$$\|\theta_{xxt}(t)\|\le C_2\left(\|\theta_x(t)\|_{H^3}+\|v_x(t)\|_{H^2}+\|u_x(t)\|_{H^2}+\|Z_x(t)\|_{H^1}\right), \tag{2.5.10}$$

$$\|Z_{xxt}(t)\|\le C_2\left(\|Z_x(t)\|_{H^3}+\|\theta_x(t)\|_{H^1}+\|v_x(t)\|_{H^2}\right) \tag{2.5.11}$$

or

$$\|u_{xxxx}(t)\|\le C_2\left(\|u_x(t)\|_{H^2}+\|v_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}+\|u_{txx}(t)\|\right), \tag{2.5.12}$$

$$\|\theta_{xxxx}(t)\|\le C_2\left(\|\theta_x(t)\|_{H^2}+\|v_x(t)\|_{H^2}+\|u_x(t)\|_{H^2}+\|Z_x(t)\|_{H^1}+\|\theta_{txx}(t)\|\right), \tag{2.5.13}$$

$$\|Z_{xxxx}(t)\| \le C_2\big(\|Z_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|Z_{txx}(t)\|\big). \tag{2.5.14}$$

Differentiating (2.2.28) with respect to t, using theorem 2.2.2, (2.3.147)–(2.3.149) and (2.5.6)–(2.5.12), we have

$$\begin{aligned}\|u_{tt}(t)\| &\le C_2\big(\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|u_{tx}(t)\| + \|u_{txx}(t)\|\big) && (2.5.15)\\ &\le C_2\big(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|Z_x(t)\|\big). && (2.5.16)\end{aligned}$$

Analogously, we can derive from (2.2.29), (2.2.30), (2.3.147)–(2.3.149) and (2.5.6)–(2.5.11) and theorem 2.2.1 that

$$\begin{aligned}\|\theta_{tt}(t)\| &\le C_2\big(\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{tx}(t)\| + \|Z_t(t)\|\big) && (2.5.17)\\ &\le C_2\big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \|Z_x(t)\|_{H^1}\big), && (2.5.18)\end{aligned}$$

$$\begin{aligned}\|Z_{tt}(t)\| &\le C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_x(t)\|_{H^1} + \|\theta_t(t)\|\\ &\qquad + \|Z_{xx}(t)\| + \|Z_{txx}(t)\| + \|Z_t(t)\|) && (2.5.19)\\ &\le C_2\big(\|Z_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|u_x(t)\|_{H^1}\big). && (2.5.20)\end{aligned}$$

Thus the estimate (2.5.2) follows from (2.5.9)–(2.5.11), (2.5.16), (2.5.18) and (2.5.20).

Differentiating (2.2.28) with respect to t twice, multiplying the resultant by u_{tt} in $L^2(0,\ L)$, performing an integration by parts, and using theorem 2.2.2 and the embedding theorem and the Young inequality, we can derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{tt}\|^2 =& -\int_0^L \left(r^{n-1}u_{tt}\right)_x \left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_{tt} dx\\ &- \int_0^L \left(\left(r^{n-1}\right)_t u_{tt}\right)_x \left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_t dx\\ &- \int_0^L \left(\left(r^{n-1}\right)_{tt} u_{tt} + \left(r^{n-1}\right)_t u_{tt}\right)_x \left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right) dx\\ \le& -\int_0^L \epsilon\frac{r_x^{2(n-1)}}{v}u_{ttx}^2 dx + C_2(\|u_{tt}\| + \|u_{xt}u_x\| + \left\|u_x^3\right\| + \|\theta_t u_x\|\\ &+ \|u_{xt}\| + \|\theta_{tt}\| + \left\|u_x^2\right\|)\|u_{ttx}\|\\ \le& -C_1^{-1}\|u_{ttx}\|^2 + C_2\Big(\|u_x\|_{H^1}^2 + \|\theta_t\|^2 + \|u_{xt}\|^2 + \|\theta_{tt}\|^2 + \|u_{tt}\|^2\Big).\end{aligned} \tag{2.5.21}$$

Thus, by theorem 2.2.2,

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{ttx}(s)\|^2 ds \le C_4 + C_4\int_0^t \Big(\|u_{tt}\|^2 + \|\theta_{tt}\|^2\Big)(s)\,ds$$

which, along with (2.5.15) and (2.5.17), gives estimate (2.5.3).

In the same manner, differentiating (2.2.29) with respect to t twice, multiplying the resultant by θ_{tt} and performing an integration by parts over $L^2(0,\, L)$, and using the embedding theorem and the Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta \theta_{tt}^2 dx\\
&\quad= -\int_0^L \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_{tt}\theta_{ttx}dx - \int_0^L \Big(e_{\theta tt}\theta_t + e_{vtt}\big(r^{n-1}u\big)_x\Big)\theta_{tt}dx - \frac{3}{2}\int_0^L e_{\theta t}\theta_{tt}^2 dx\\
&\qquad -\int_0^L \left(e_v + P - \epsilon\frac{(r^{n-1}u)_x}{v}\right)\big(r^{n-1}u\big)_{xtt}\theta_{tt}dx + \int_0^L \left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_{tt}\big(r^{n-1}u\big)_x\theta_{tt}dx\\
&\qquad -2\int_0^L \left(e_{vt} + \left(P - \epsilon\frac{(r^{n-1}u)_x}{v}\right)_t\right)\big(r^{n-1}u\big)_{xt}\theta_{tt}dx - 2\mu(n-1)\int_0^L \big(r^{n-2}u^2\big)_{xtt}\theta_{tt}dx\\
&\qquad + k\lambda\int_0^L (\phi Z)_{tt}\theta_{tt}dx \;=: \sum_{i=1}^{8} A_i.
\end{aligned}\tag{2.5.22}$$

By virtue of theorems 2.2.1, 2.2.2 and lemmas 2.4.1–2.4.6, and using the embedding theorem, we derive that for any $\varepsilon \in (0,\, 1)$,

$$\begin{aligned}
A_1 \le &-C_1\|\theta_{ttx}\|^2 + C_2(\|\theta_x\|_{L^\infty}\|u_{xt}\| + \|u_x\|_{L^\infty}\|\theta_{xt}\| + \|u_x\|_{L^\infty}^2\|\theta_x\|\\
&+\|\theta_x\|_{L^\infty}\|\theta_t\| + \|\theta_x\|_{L^\infty}\|\theta_{tt}\|)\|\theta_{ttx}\|\\
\le &-(2C_1)^{-1}\|\theta_{ttx}\|^2 + C_2\Big(\|\theta_{xt}\|^2 + \|u_{xt}\|^2 + \|u_x\|_{H^1}^2 + \|\theta_{tt}\|^2\Big),
\end{aligned}\tag{2.5.23}$$

$$\begin{aligned}
A_2 &\le C_1\int_0^L \Big((|u_x| + |\theta_t|)^2 + |u_{xt}| + |\theta_{tt}|\Big)(|u_x| + |\theta_t|)|\theta_{tt}|dx\\
&\le C_1\|\theta_{tt}\|_{L^\infty}(\|u_x\| + \|\theta_t\|)\big((\|u_x\|_{L^\infty} + \|\theta_t\|_{L^\infty})(\|u_x\| + \|\theta_t\|) + \|u_{xt}\| + \|\theta_{tt}\|\big)\\
&\le C_2(\|\theta_{tt}\| + \|\theta_{ttx}\|)\big(\|u_x\|_{H^1} + \|\theta_t\| + \|\theta_{xt}\| + \|u_{xt}\| + \|\theta_{tt}\|\big)\\
&\le \varepsilon\|\theta_{ttx}\|^2 + C_2\varepsilon^{-1}\Big(\|u_x\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{xt}\|^2 + \|u_{xt}\|^2 + \|\theta_{tt}\|^2\Big),
\end{aligned}\tag{2.5.24}$$

$$\begin{aligned}
A_3 &\le C_1\int_0^L (|u_x| + |\theta_t|)\theta_{tt}^2 dx \le C_1\|\theta_{tt}\|_{L^\infty}(\|u_x\| + \|\theta_t\|)\|\theta_{tt}\|\\
&\le C_1(\|\theta_{tt}\| + \|\theta_{ttx}\|)(\|u_x\| + \|\theta_t\|)\|\theta_{tt}\| \le \varepsilon\|\theta_{ttx}\|^2 + C_2\varepsilon^{-1}\|\theta_{tt}\|^2,
\end{aligned}\tag{2.5.25}$$

$$A_4 \le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\|\theta_{tt}\|^2,\tag{2.5.26}$$

$$\begin{aligned}A_5 &\le C_2\|u_x\|_{L^\infty}\|\theta_{tt}\|\big(\big(\|u_x\|_{L^\infty}+\|\theta_t\|_{L^\infty}\big)(\|u_x\|+\|\theta_t\|)+\|u_{xt}\| \\ &\quad +\|\theta_{tt}\|+\|u_{xtt}\|+\|u_{tt}\|+\|u_x\|) \\ &\le C_2\|\theta_{tt}\|\big(\|u_x\|_{H^1}+\|\theta_t\|+\|\theta_{xt}\|+\|u_{xt}\|+\|\theta_{tt}\|+\|u_{xtt}\|+\|u_{tt}\|\big) \\ &\le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\Big(\|\theta_{tt}\|^2+\|u_x\|^2_{H^1}+\|\theta_t\|^2+\|\theta_{xt}\|^2+\|u_{xt}\|^2\Big),\end{aligned} \tag{2.5.27}$$

$$\begin{aligned}A_6 &\le C_1\int_0^L(|u_x|+|\theta_t|+|u_{xt}|+|u_x|^2+|u_t|)(|u_{xt}|+|u_t|)|\theta_{tt}|\,dx \\ &\le C_2\|u_{tx}\|^{\frac12}\|u_{txx}\|^{\frac12}(\|u_x\|+\|\theta_t\|+\|u_{xt}\|)\|\theta_{tt}\|,\end{aligned} \tag{2.5.28}$$

$$\begin{aligned}A_7 &\le C_2\|\theta_{tt}\|\big(\|u_t\|+\|u_x\|+\|u_{xt}\|+\|u_x\|_{L^\infty}\|u_t\|+\|u_x\|_{L^\infty}\|u_{tt}\|+\|u_t\|_{L^\infty}\|u_{xt}\|+\|u_{xtt}\|\big) \\ &\le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\Big(\|u_x\|^2_{H^1}+\|u_{xt}\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2\Big),\end{aligned} \tag{2.5.29}$$

$$\begin{aligned}A_8 &\le C_1\int_0^L(|\theta_t Z_t|+|\theta_{tt}|+|Z_{tt}|)|\theta_{tt}|\,dx \le C_1(\|\theta_t Z_t\|+\|\theta_{tt}\|+\|Z_{tt}\|)\|\theta_{tt}\| \\ &\le \varepsilon\Big(\|\theta_{ttx}\|^2+\|Z_{txx}\|^2\Big)+C_2\varepsilon^{-1}\Big(\|\theta_{tt}\|^2+\|Z_t\|^2_{H^1}+\|\theta_t\|^2_{H^1}\Big).\end{aligned} \tag{2.5.30}$$

It follows from (2.5.28), by the Hölder inequality, that

$$\begin{aligned}\int_0^t A_6\,ds &\le C_2\sup_{0\le s\le t}\bigg(\int_0^t\|u_{txx}(s)\|^2\,ds\bigg)^{\frac14}\bigg(\int_0^t\|u_{tx}(s)\|^2\,ds\bigg)^{\frac14} \\ &\quad\times\bigg(\int_0^t\Big(\|u_x\|^2+\|\theta_t\|^2+\|u_{tx}\|^2\Big)(s)\,ds\bigg)^{\frac12} \\ &\le \varepsilon\bigg(\sup_{0\le s\le t}\|\theta_{tt}(s)\|^2+\int_0^t\|u_{txx}(s)\|^2\,ds\bigg)+C_2\varepsilon^{-3}.\end{aligned} \tag{2.5.31}$$

Thus we derive from (2.5.22)–(2.5.31) that for any $\varepsilon \in (0,\,1)$ small enough,

$$\begin{aligned}&\|\theta_{tt}(t)\|^2+\int_0^t\|\theta_{ttx}(s)\|^2\,ds \\ &\le C_4\varepsilon^{-3}+C_2\varepsilon^{-1}\int_0^t\|\theta_{tt}(s)\|^2\,ds \\ &\quad + C_1\varepsilon\bigg(\sup_{0\le s\le t}\|\theta_{tt}(s)\|^2+\int_0^t\Big(\|u_{txx}\|^2+\|u_{ttx}\|^2+\|Z_{txx}\|^2\Big)(s)\,ds\bigg).\end{aligned} \tag{2.5.32}$$

Therefore taking supremum in t on the left-hand side of (2.5.32), picking $\varepsilon \in (0,\,1)$ small enough, and using (2.5.17), we can derive estimate (2.5.4).

Differentiating (2.2.30) with respect to t twice, multiplying the resultant by Z_{tt} in $L^2(0,\,L)$ and integrating by parts, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_{tt}(t)\|^2 &= -\int_0^L \left(\frac{dr^{2n-2}Z_x}{v^2}\right)_{tt} Z_{ttx}\,dx - k\lambda\int_0^L (\phi Z)_{tt} Z_{tt}\,dx\\ &\le -\int_0^L \frac{dr^{2n-2}Z_{xtt}^2}{v^2}dx + C_2((\|Z_{xt}\|+\|u_{xt}\|+\|Z_x\|)\|Z_{xtt}\|\\ &\quad +\|Z_{tt}\|\big(\|\theta_{tt}\|+\|Z_{tt}\|+\|u_x\|_{H^1}+\|\theta_t\|_{H^1}))\\ &\le -(2C_1)^{-1}\|Z_{ttx}\|^2 + C_2(\|Z_{tt}\|^2+\|\theta_{tt}\|^2+\|u_{xt}\|^2+\|u_x\|_{H^1}^2\\ &\quad +\|\theta_t\|_{H^1}^2+\|Z_t\|_{H^1}^2+\|Z_x\|^2). \end{aligned} \tag{2.5.33}$$

Thus, by lemmas 2.4.1–2.4.6 and (2.2.28),

$$\|Z_{tt}(t)\|^2 + \int_0^t \|Z_{xtt}(s)\|^2 ds \le C_4 + C_2\int_0^t \Big(\|Z_{tt}\|^2+\|\theta_{tt}\|^2\Big)(s)\,ds \tag{2.5.34}$$

which, together with (2.5.17) and (2.5.19), gives (2.5.5). The proof is complete. □

Secondly, we establish estimates on u_{xt}, θ_{xt} and Z_{xt}.

Lemma 2.5.2. *Under assumptions of theorem 2.2.3, the following estimates hold that for any $t > 0$ and for $\varepsilon > 0$ small enough,*

$$\|u_{xt}(t)\|^2 + \int_0^t \|u_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2\int_0^t \Big(\|u_{xtt}\|^2+\|\theta_{xxt}\|^2\Big)(s)\,ds, \tag{2.5.35}$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2\int_0^t \|u_{xxt}(s)\|^2 ds, \tag{2.5.36}$$

$$\|Z_{xt}(t)\|^2 + \int_0^t \|Z_{xxt}(s)\|^2 ds \le C_4. \tag{2.5.37}$$

Proof. Differentiating (2.2.28) with respect to x and t, multiplying the resulting equation by u_{xt} in $L^2(0, L)$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{xt}\|^2 = B_0(t) + B_1(t) \tag{2.5.38}$$

with

$$\begin{aligned} B_0(t) &= \left(r^{n-1}\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_x\right)_t u_{xt}\Big|_{x=0}^{x=L},\\ B_1(t) &= -\int_0^L \left(r^{n-1}\left(\epsilon\frac{(r^{n-1}u)_x}{v} - P\right)_x\right)_t u_{xxt}\,dx. \end{aligned}$$

Employing lemmas 2.4.1–2.4.3 and lemma 2.5.1, the interpolation inequality and the Poincaré inequality, we can get

$$\begin{aligned}B_0(t) \le{}& C_1\big(\big(\|u_x\|_{L^\infty}+\|\theta_x\|_{L^\infty}\big)\big(\|u_x\|_{L^\infty}+\|\theta_x\|_{L^\infty}+\|v_x\|_{L^\infty}\big)+\|u_{xx}\|_{L^\infty}+\|\theta_{xt}\|_{L^\infty}\\
&+\|u_{xxt}\|_{L^\infty}+\|v_x\|_{L^\infty}\|u_{xt}\|_{L^\infty}+\|u_x\|_{L^\infty}\|u_{xx}\|_{L^\infty}+\big\|u_x^2\big\|_{L^\infty}+\|v_{xt}\|_{L^\infty}\\
&+\|v_x\|_{L^\infty}\|\theta_t\|_{L^\infty}+\|u_x\|_{L^\infty}\|\theta_x\|_{L^\infty}+\|\theta_x\|_{L^\infty}\|\theta_t\|_{L^\infty}+\|u_x\|_{L^\infty}\|v_x\|_{L^\infty})\|u_{xt}\|_{L^\infty}\\
\le{}& C_2(B_{01}+B_{02})\|u_{xt}\|^{\frac{1}{2}}\|u_{xxt}\|^{\frac{1}{2}}\end{aligned} \tag{2.5.39}$$

where

$$B_{01}=\|u_x\|_{H^2}+\|\theta_t\|+\|\theta_{xt}\|$$

and

$$B_{02}=\|\theta_{xt}\|^{\frac{1}{2}}\|\theta_{xxt}\|^{\frac{1}{2}}+\|u_{xxt}\|^{\frac{1}{2}}\|u_{xxxt}\|^{\frac{1}{2}}+\|u_{xxt}\|+\|u_{xt}\|^{\frac{1}{2}}\|u_{xxt}\|^{\frac{1}{2}}.$$

Applying the Young inequality several times, we have that for any $\varepsilon\in(0,\,1)$,

$$C_2B_{01}\|u_{xt}\|^{\frac{1}{2}}\|u_{xxt}\|^{\frac{1}{2}}\le\frac{\varepsilon^2}{2}\|u_{xxt}\|^2+C_2\varepsilon^{-1}\Big(\|u_x\|_{H^2}^2+\|\theta_t\|_{H^1}^2+\|u_{xt}\|^2\Big) \tag{2.5.40}$$

and

$$C_2B_{02}\|u_{xt}\|^{\frac{1}{2}}\|u_{xxt}\|^{\frac{1}{2}}\le\frac{\varepsilon^2}{2}\|u_{xxt}\|^2+\varepsilon^2\Big(\|\theta_{txx}\|^2+\|u_{xxxt}\|^2\Big)+C_2\varepsilon^{-6}\Big(\|\theta_{tx}\|^2+\|u_{xt}\|^2\Big). \tag{2.5.41}$$

Thus we derive from (2.5.39)–(2.5.41) and lemmas 2.3.1–2.3.13 and 2.5.1 that

$$B_0\le\varepsilon^2\Big(\|u_{xxt}\|^2+\|\theta_{txx}\|^2+\|u_{xxxt}\|^2\Big)+C_2\varepsilon^{-6}\Big(\|\theta_x\|^2+\|u_x\|_{H^2}^2+\|\theta_{tx}\|^2+\|u_{xt}\|^2\Big) \tag{2.5.42}$$

which, together with theorem 2.2.2, further leads to

$$\int_0^t B_0\,ds\le\varepsilon^2\int_0^t\Big(\|u_{xxt}\|^2+\|\theta_{txx}\|^2+\|u_{xxxt}\|^2\Big)(s)\,ds+C_2\varepsilon^{-6}. \tag{2.5.43}$$

Analogously, by lemma 2.5.1 and theorem 2.2.1 and the embedding theorem, we can get that for any $\varepsilon\in(0,\,1)$,

$$\begin{aligned}B_1\le{}&-\int_0^L\epsilon\frac{r^{2n-2}}{v}u_{txx}^2\,dx+C_1((\|u_x\|+\|\theta_t\|+\|v_x\|)\big(\|\theta_x\|_{L^\infty}+\|v_x\|_{L^\infty}\big)\\
&+\|u_{xx}\|+\|\theta_{xt}\|+\|v_x\|_{L^\infty}\|u_{xt}\|+\|u_x\|_{L^\infty}\|u_{xx}\|+\|u_x\|_{L^\infty}^2\|v_x\|\\
&+\|u_x\|+\|\theta_x\|+\|u_x\|^2)\|u_{xxt}\|\le-(2C_1)^{-1}\|u_{xxt}\|^2\\
&+C_2\Big(\|u_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|u_{xt}\|^2+\|v_x\|^2\Big)\end{aligned} \tag{2.5.44}$$

which, combined with (2.5.38), (2.5.43) and theorem 2.2.2, gives that for any $\varepsilon\in(0,\,1)$ small enough,

$$\|u_{xt}(t)\|^2+\int_0^t\|u_{xxt}(s)\|^2ds\le C_2\varepsilon^{-6}+C_1\varepsilon^2\int_0^t\Big(\|\theta_{txx}\|^2+\|u_{xxxt}\|^2\Big)(s)\,ds. \tag{2.5.45}$$

On the other hand, differentiating (2.2.28) with respect to x and t, and using theorem 2.2.2 and lemma 2.5.1, we can derive

$$\|u_{xxxt}(t)\|\le C_1\|u_{xtt}(t)\|+C_2\big(\|u_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^1}+\|v_x(t)\|_{H^1}+\|\theta_t(t)\|_{H^2}\big). \tag{2.5.46}$$

Thus inserting (2.5.46) into (2.5.45) implies (2.5.35).

Similarly, we derive from (2.2.29) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tx}^2dx=:\sum_{i=1}^5 D_i(t) \tag{2.5.47}$$

where

$$D_1(t)=-\int_0^L\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_{tx}\theta_{txx}dx,\ D_2(t)=-\int_0^L\left(\left(e_v+P-\epsilon\frac{(r^{n-1}u)_x}{v}\right)\left(r^{n-1}u\right)_x\right)_{xt}\theta_{tx}dx,$$
$$D_3(t)=-\int_0^L\left(e_{\theta tx}\theta_t+\frac{1}{2}e_{\theta t}\theta_{tx}+e_{\theta x}\theta_{tt}\right)\theta_{tx}dx,\ D_4(t)=2\mu(n-1)\int_0^L\left(r^{n-2}u^2\right)_{xt}\theta_{txx}dx,$$
$$D_5(t)=k\lambda\int_0^L(\phi Z)_{tx}\theta_{tx}dx.$$

By virtue of lemmas 2.3.13, 2.5.1, and (2.5.35), and using the embedding theorem and the Young inequality, we derive that for any $\varepsilon\in(0,1)$,

$$D_1(t)\le-(2C_1)^{-1}\|\theta_{txx}\|^2+C_2\Big(\|u_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2\Big), \tag{2.5.48}$$

$$D_2(t)\le\varepsilon^2\|u_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^2}^2+\|\theta_t\|_{H^1}^2+\|u_{xt}\|^2+\|v_x\|_{H^1}^2\Big), \tag{2.5.49}$$

$$D_3(t)\le\varepsilon^2\|\theta_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|\theta_x\|_{H^2}^2+\|u_{xt}\|^2+\|v_x\|^2\Big), \tag{2.5.50}$$

$$D_4(t)\le\varepsilon^2\|\theta_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^1}^2+\|u_t\|^2+\|u_{xt}\|^2\Big), \tag{2.5.51}$$

$$D_5(t)\le C_2\Big(\|Z_x\|_{H^1}^2+\|Z_{xt}\|^2+\|\theta_{xt}\|^2\Big) \tag{2.5.52}$$

which, along with (2.5.47)–(2.5.51), implies (2.5.36).

Analogously, differentiating (2.2.30) with respect to t and x, multiplying the resultant by Z_{xt} and performing an integration by parts, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_{xt}\|^2 &= -\int_0^L \left(\frac{dr^{2n-2}Z_x}{v^2}\right)_{tx} Z_{txx}\,dx - k\lambda\int_0^L (\phi Z)_{tx} Z_{tx}\,dx\\ &\le -d\int_0^L \frac{r^{2n-2}Z_{xxt}^2}{v^2}\,dx + C_1\big((\|Z_{xt}\| + \|u_x\|_{H^1} + \|Z_{xx}\|)\|Z_{xxt}\|\\ &\quad + \|Z_{xt}\|(\|\theta_{xt}\| + \|Z_{xt}\| + \|Z_{xx}\|)\big)\\ &\le -(2C_1)^{-1}\|Z_{txx}\|^2 + C_2\left(\|Z_{xt}\|^2 + \|Z_{xx}\|^2 + \|\theta_{xt}\|^2 + \|u_x\|^2_{H^1}\right) \end{aligned} \tag{2.5.53}$$

which, together with lemmas 2.3.10, 2.4.1, 2.4.3 and 2.5.1, gives (2.5.37). □

Finally, we turn to show estimates on u, v, θ, Z in H^4.

Lemma 2.5.3. *Under assumptions of theorem 2.2.3, we have for any $t > 0$,*

$$\begin{aligned}&\|u_{tt}\|^2 + \|u_{xt}\|^2 + \|\theta_{tt}\|^2 + \|\theta_{xt}\|^2 + \|Z_{tt}\|^2 + \|Z_{xt}\|^2\\ &\quad + \int_0^t \left(\|u_{ttx}\|^2 + \|u_{xxt}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{xxt}\|^2 + \|Z_{ttx}\|^2 + \|Z_{xxt}\|^2\right)(s)\,ds \le C_4,\end{aligned} \tag{2.5.54}$$

$$\begin{aligned}&\|v_{xxx}\|^2_{H^1} + \|u_{xxx}\|^2_{H^1} + \|\theta_{xxx}\|^2_{H^1} + \|Z_{xxx}\|^2_{H^1} + \|u_{txx}\|^2 + \|\theta_{txx}\|^2 + \|Z_{txx}\|^2\\ &\quad + \int_0^t \left(\|u_{tt}\|^2 + \|u_{xxt}\|^2_{H^1} + \|\theta_{tt}\|^2 + \|\theta_{xxt}\|^2_{H^1} + \|Z_{tt}\|^2 + \|Z_{xxt}\|^2_{H^1}\right)(s)\,ds \le C_4,\end{aligned} \tag{2.5.55}$$

$$\int_0^t \left(\|v_{xxx}\|^2_{H^1} + \|u_{xxxx}\|^2_{H^1} + \|\theta_{xxxx}\|^2_{H^1} + \|Z_{xxxx}\|^2_{H^1}\right)(s)\,ds \le C_4. \tag{2.5.56}$$

Proof. Adding up (2.5.35)–(2.5.37) and choosing $\varepsilon > 0$ small enough, we conclude

$$\begin{aligned}&\|u_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|Z_{xt}(t)\|^2 + \int_0^t \left(\|u_{xxt}\|^2 + \|\theta_{xxt}\|^2 + \|Z_{xxt}\|^2\right)(s)\,ds\\ &\quad \le C_4 + C_2\varepsilon^2\int_0^t \|u_{xtt}(s)\|^2\,ds.\end{aligned} \tag{2.5.57}$$

Now multiplying (2.5.3), (2.5.4) and (2.5.5) by ε, ε^2 and ε, respectively, then adding up the resultants to (2.5.57) and picking ε small sufficiently, we obtain (2.5.54).

Differentiating (2.2.28) with respect to x and noting that $v_{xxt} = (r^{n-1}u)_{xxx}$, we get

$$\epsilon\frac{\partial}{\partial t}\left(\frac{v_{xx}}{v}\right) - P_v v_{xx} = r^{1-n}u_{tx} + E(x,t) + (1-n)r^{1-2n}vu_t \tag{2.5.58}$$

where

$$\begin{aligned}E(x,t) &= P_{vv}v_x^2 + 2P_{v\theta}\theta_x v_x + 2P_{\theta\theta}\theta_x^2 + P_\theta\theta_{xx} - 2\epsilon\frac{v_x^2}{v^3}\left(r^{n-1}u\right)_x + 2\epsilon\frac{v_x}{v^2}\left(r^{n-1}u\right)_{xx}\\ &= 2R\frac{\theta v_x^2}{v^3} - 2R\frac{\theta_x v_x}{v^2} + 8a\theta^2\theta_x^2 + \left(\frac{R}{v} + \frac{4}{3}a\theta^3\right)\theta_{xx} - 2\epsilon\left(\frac{v_x}{v^2}\left(r^{n-1}u\right)_{xx} - \frac{v_x^2}{v^3}\left(r^{n-1}u\right)_x\right).\end{aligned}$$

Differentiating (2.5.58) with respect to x, we can get

$$\epsilon\frac{\partial}{\partial t}\left(\frac{v_{xxx}}{v}\right) - P_v v_{xxx} = E_1(x,t) \tag{2.5.59}$$

where

$$E_1(x,t) = E_x(x,t) + P_{vx}v_{xx} + \epsilon\left(\frac{v_{xx}v_x}{v^2}\right)_t + r^{1-n}u_{txx} + 3(1-n)r^{1-2n}vu_{tx}$$
$$+ (1-n)(1-2n)r^{1-3n}v^2u_t + (1-n)r^{1-2n}v_xu_t.$$

Obviously, we can derive from theorem 2.2.2 and lemmas 2.5.1 and 2.5.2 that

$$\|E_1(t)\| \le C_2\big(\|v_x\|_{H^1} + \|u_x\|_{H^2} + \|\theta_x\|_{H^2} + \|u_{txx}\|\big) \tag{2.5.60}$$

and

$$\int_0^t \|E_1(s)\|^2 ds \le C_4. \tag{2.5.61}$$

Multiplying (2.5.59) by $\frac{v_{xxx}}{v}$ over $L^2(0,\,L)$, we can obtain

$$\frac{d}{dt}\left\|\frac{v_{xxx}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{xxx}}{v}\right\|^2 \le C_1\|E_1(t)\|^2 \tag{2.5.62}$$

which, combined with (2.5.61), gives

$$\|v_{xxx}(t)\|^2 + \int_0^t \|v_{xxx}(s)\|^2 ds \le C_4. \tag{2.5.63}$$

Noting (2.4.39), (2.4.41), (2.4.43), (2.5.12)–(2.5.14), (2.5.54) and (2.5.63), and using the embedding theorem, theorem 2.2.2 and lemmas 2.5.1 and 2.5.2, we can derive for any $t > 0$,

$$\|u_{xxx}\|^2 + \|\theta_{xxx}\|^2 + \|Z_{xxx}\|^2 + \|u_{xx}\|^2_{L^\infty} + \|\theta_{xx}\|^2_{L^\infty} + \|Z_{xx}\|^2_{L^\infty} + \int_0^t (\|u_{xxx}\|^2_{H^1} + \|\theta_{xxx}\|^2_{H^1}$$
$$+ \|Z_{xxx}\|^2_{H^1} + \|u_{xx}\|^2_{W^{1,\infty}} + \|\theta_{xx}\|^2_{W^{1,\infty}} + \|Z_{xx}\|^2_{W^{1,\infty}})(s)\,ds \le C_4. \tag{2.5.64}$$

Differentiating (2.2.28)–(2.2.30) with respect to t and using (2.5.54), (2.5.63), (2.5.64), theorem 2.2.2 and lemmas 2.5.1 and 2.5.2, we can conclude

$$\|u_{txx}\| \le C_1\|u_{tt}\| + C_2\big(\|u_x\|_{H^1} + \|v_x\| + \|\theta_x\| + \|\theta_t\| + \|\theta_{xt}\| + \|u_{xt}\|\big) \le C_4, \tag{2.5.65}$$

$$\|\theta_{txx}\| \le C_1\|\theta_{tt}\| + C_2\big(\|u_x\|_{H^1} + \|v_x\| + \|\theta_x\|_{H^2} + \|\theta_t\|_{H^1} + \|u_{xt}\| + \|Z_t\|\big) \le C_4, \tag{2.5.66}$$

$$\|Z_{txx}\| \le C_1\|Z_{tt}\| + C_2\big(\|Z_{xx}\| + \|u_x\|_{H^1} + \|\theta_x\|_{H^1} + \|v_x\| + \|\theta_t\| + \|Z_t\|\big) \le C_4 \tag{2.5.67}$$

which, combined with (2.5.12)–(2.5.14) and (2.5.64)–(2.5.66), implies

$$\|u_{xxxx}(t)\|^2+\|\theta_{xxxx}(t)\|^2+\|Z_{xxxx}(t)\|^2+\int_0^t(\|u_{txx}\|^2+\|\theta_{txx}\|^2+\|Z_{txx}\|^2 \\ +\|u_{xxxx}\|^2+\|\theta_{xxxx}\|^2+\|Z_{xxxx}\|^2)(s)\,ds\le C_4. \tag{2.5.68}$$

Therefore it follows from (2.5.64), (2.5.68) and the embedding theorem that

$$\|u_{xxx}(t)\|_{L^\infty}^2+\|\theta_{xxx}(t)\|_{L^\infty}^2+\|Z_{xxx}(t)\|_{L^\infty}^2 \\ +\int_0^t\Big(\|u_{xxx}\|_{L^\infty}^2+\|\theta_{xxx}\|_{L^\infty}^2+\|Z_{xxx}\|_{L^\infty}^2\Big)(s)\,ds\le C_4. \tag{2.5.69}$$

Now differentiating (2.5.59) with respect to x, we find

$$\epsilon\frac{\partial}{\partial t}\Big(\frac{v_{xxxx}}{v}\Big)-P_v v_{xxxx}=E_2(x,t) \tag{2.5.70}$$

where

$$E_2(x,t)=E_{1x}(x,t)+P_{vx}v_{xxx}+\epsilon\Big(\frac{v_{xxx}v_x}{v^2}\Big)_t.$$

Using the embedding theorem, lemmas 2.5.1 and 2.5.2 and (2.5.64)–(2.5.69), we derive that

$$\begin{aligned}\|E_{xx}(t)\|&\le C_4\big(\|u_x\|_{H^3}+\|\theta_x\|_{H^3}+\|v_x\|_{H^2}\big),\\ \|E_{1x}(t)\|&\le C_1\Bigg(\|E_{xx}(t)\|+\|u_{xxxt}\|+\|u_{xxt}\|+\big\|(P_{vx}v_{xx})_x\big\|+\|v_xu_{xt}\|\\ &\qquad+\|v_xu_t\|+\|v_{xx}u_t\|+\Big\|\Big(\frac{v_xv_{xx}}{v^2}\Big)_{xt}\Big\|\Bigg)\\ &\le C_1\|u_{xxxt}\|+C_4\big(\|u_x\|_{H^3}+\|\theta_x\|_{H^3}+\|v_x\|_{H^2}\big)\end{aligned}$$

whence

$$\|E_2(t)\|\le C_1\|u_{xxxt}\|+C_4\big(\|u_x\|_{H^3}+\|\theta_x\|_{H^3}+\|v_x\|_{H^2}\big). \tag{2.5.71}$$

We derive from (2.5.15)–(2.5.20) that

$$\int_0^t\Big(\|u_{tt}\|^2+\|\theta_{tt}\|^2+\|Z_{tt}\|^2\Big)(s)\,ds\le C_4 \tag{2.5.72}$$

which, together with (2.5.46) and (2.5.54), gives

$$\int_0^t\|u_{xxxt}(s)\|^2\,ds\le C_4. \tag{2.5.73}$$

Thus it follows from (2.5.64), (2.5.68), (2.5.71), (2.5.73), theorem 2.2.2 and lemmas 2.5.1 and 2.5.2 that

$$\int_0^t \|E_2(s)\|^2 ds \leq C_4. \tag{2.5.74}$$

Multiplying (2.5.70) by $\frac{v_{xxxx}}{v}$ in $L^2(0, L)$, we can get

$$\frac{d}{dt}\left\|\frac{v_{xxxx}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{xxxx}}{v}\right\|^2 \leq C_1\|E_2(t)\|^2 \tag{2.5.75}$$

whence, by (2.5.74),

$$\|v_{xxxx}(t)\|^2 + \int_0^t \|v_{xxxx}(s)\|^2 ds \leq C_4. \tag{2.5.76}$$

Differentiating (2.2.29) and (2.2.30) with respect to x and t, respectively, and using theorem 2.2.2 and lemmas 2.5.1 and 2.5.2, we can derive

$$\|\theta_{txxx}\| \leq C_1\|\theta_{ttx}\| + C_2\big(\|u_x\|_{H^3} + \|v_x\|_{H^2} + \|\theta_x\|_{H^3} + \|\theta_{xt}\| + \|Z_{xt}\|\big), \tag{2.5.77}$$

$$\|Z_{txxx}\| \leq C_1\|Z_{ttx}\| + C_2\big(\|Z_x\|_{H^2} + \|u_x\|_{H^3} + \|v_x\|_{H^3} + \|\theta_{xt}\| + \|Z_{xt}\|\big). \tag{2.5.78}$$

Thus,

$$\int_0^t \left(\|\theta_{txxx}\|^2 + \|Z_{txxx}\|^2\right)(s)\, ds \leq C_4. \tag{2.5.79}$$

Differentiating (2.2.28) with respect to x three times, using lemmas 2.5.1 and 2.5.2 and theorem 2.2.2 and applying the Poincaré inequality, we derive

$$\|u_{xxxxx}\| \leq C_1\|u_{txxx}\| + C_2\big(\|u_x\|_{H^3} + \|v_x\|_{H^3} + \|\theta_x\|_{H^3}\big). \tag{2.5.80}$$

Thus we derive from (2.5.63), (2.5.64), (2.5.68), (2.5.69), (2.5.73) and (2.5.76) that

$$\int_0^t \|u_{xxxxx}(s)\|^2 ds \leq C_4. \tag{2.5.81}$$

Similarly, we can derive from (2.2.29), (2.2.30), (2.5.68), (2.5.76) and (2.5.79) that

$$\int_0^t \left(\|\theta_{xxxxx}\|^2 + \|Z_{xxxxx}\|^2\right)(s)\, ds \leq C_4. \tag{2.5.82}$$

Hence, (2.5.55) and (2.5.56) follow from (2.5.63)–(2.5.82). □

By lemmas 2.5.1–2.5.3, we have proved the global existence of solutions to the problem (2.2.27)–(2.2.32) in $\mathcal{H}^4$ with arbitrary initial datum $(u_0, v_0, \theta_0, Z_0) \in \mathcal{H}^4$ and the uniqueness of solutions in $\mathcal{H}^4$ follows from that of global solutions in $\mathcal{H}^1$ or in $\mathcal{H}^2$.

2.5.2 Exponential Stability of Solutions in $\mathcal{H}^4$

In this subsection, we shall use estimates established in lemmas 2.5.1–2.5.3 to prove the exponential stability of solutions on $\mathcal{H}^4$. We first estimate $\|u_{tt}(t)\|$, $\|\theta_{tt}(t)\|$ and $\|Z_{tt}(t)\|$.

Lemma 2.5.4. *Under assumptions of theorem 2.2.3, for any* $(u_0, v_0, \theta_0, Z_0) \in \mathcal{H}^4$, *there exists a positive constant* $\gamma_4^{(1)} = \gamma_4^{(1)}(C_4) \le \gamma_2(C_2)$ *such that for any fixed* $\gamma \in \left(0, \gamma_4^{(1)}\right]$, *the following estimates hold for any* $t > 0$ *and* $\varepsilon \in (0, 1)$ *small enough,*

$$e^{\gamma t}\|u_{tt}(t)\|^2 + \int_0^t e^{\gamma s}\|u_{ttx}(s)\|^2 ds \le C_4 + C_2\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2 ds, \tag{2.5.83}$$

$$\begin{aligned} &e^{\gamma t}\|\theta_{tt}(t)\|^2 + \int_0^t e^{\gamma s}\|\theta_{ttx}(s)\|^2 ds \\ &\quad \le C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t e^{\gamma s}\left(\|\theta_{txx}\|^2 + \|Z_{txx}\|^2\right)(s)\,ds \\ &\qquad + \varepsilon\int_0^t e^{\gamma s}\left(\|u_{txx}\|^2 + \|u_{ttx}\|^2\right)(s)\,ds, \end{aligned} \tag{2.5.84}$$

$$e^{\gamma t}\|Z_{tt}(t)\|^2 + \int_0^t e^{\gamma s}\|Z_{ttx}(s)\|^2 ds \le C_4 + C_2\int_0^t e^{\gamma s}\left(\|\theta_{txx}\|^2 + \|Z_{txx}\|^2\right)(s)\,ds. \tag{2.5.85}$$

Proof. Multiplying (2.5.21) by $e^{\gamma t}$ and using (2.5.17) and theorem 2.2.2, we obtain

$$\begin{aligned} \frac{1}{2}e^{\gamma t}\|u_{tt}(t)\|^2 &\le C_4 - \left(C_1^{-1} - C_1\gamma\right)\int_0^t e^{\gamma s}\|u_{ttx}(s)\|^2 ds + C_2\int_0^t e^{\gamma s}\|\theta_{tt}(s)\|^2 ds \\ &\le C_4 - \left(C_1^{-1} - C_1\gamma\right)\int_0^t e^{\gamma s}\|u_{ttx}(s)\|^2 ds + C_4\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2 ds \end{aligned}$$

which gives (2.5.83) if we take $\gamma > 0$ so small that $0 < \gamma \le \min\left(\frac{1}{4C_1^2}, \gamma_2(C_2)\right)$. Similarly, (2.5.84) and (2.5.85) can be obtained from (2.5.22), (2.5.33) and theorem 2.2.2. □

Next, we give the following estimates on $\|u_{xt}(t)\|$, $\|\theta_{xt}(t)\|$ and $\|Z_{xt}(t)\|$.

Lemma 2.5.5. *Under assumptions of theorem 2.2.3, for any* $(u_0, v_0, \theta_0, Z_0) \in \mathcal{H}^4$, *there exists a positive constant* $\gamma_4^{(2)} = \gamma_4^{(2)}(C_4) \le \gamma_4^{(1)}$ *such that for any fixed* $\gamma \in \left(0, \gamma_4^{(2)}\right]$, *the following estimates hold for any* $t > 0$ *and for* $\varepsilon \in (0, 1)$ *small enough,*

$$e^{\gamma t}\|u_{xt}(t)\|^2+\int_0^t e^{\gamma s}\|u_{txx}(s)\|^2\,ds\le C_4+C_2\varepsilon^2\int_0^t e^{\gamma s}\Big(\|u_{ttx}\|^2+\|\theta_{txx}\|^2\Big)(s)\,ds, \quad (2.5.86)$$

$$e^{\gamma t}\|\theta_{xt}(t)\|^2+\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2\,ds\le C_4+C_2\varepsilon^2\int_0^t e^{\gamma s}\Big(\|\theta_{ttx}\|^2+\|u_{txx}\|^2\Big)(s)\,ds, \quad (2.5.87)$$

$$e^{\gamma t}\|Z_{xt}(t)\|^2+\int_0^t e^{\gamma s}\|Z_{txx}(s)\|^2\,ds\le C_4. \quad (2.5.88)$$

Proof. Multiplying (2.5.38) by $e^{\gamma t}$ and using (2.5.43), (2.5.44) and theorem 2.2.1, we derive that for any $\varepsilon\in(0,1)$ small enough,

$$\begin{aligned}\frac{1}{2}e^{\gamma t}\|u_{xt}(t)\|^2\le{}& C_4-\Big((2C_1)^{-1}-\varepsilon^2\Big)\int_0^t e^{\gamma s}\|u_{txx}(s)\|^2\,ds\\&+\varepsilon^2\int_0^t e^{\gamma s}\Big(\|u_{txxx}\|^2+\|\theta_{txx}\|^2\Big)(s)\,ds\end{aligned}$$

which, together with (2.5.46), gives (2.5.86) if we take $\gamma>0$ and $\varepsilon\in(0,1)$ so small that $0<\gamma\le\min\Big(\frac{1}{8C_1^2},\gamma_4^{(1)}\Big)\equiv\gamma_4^{(2)}$ and $0<\varepsilon\le\min\Big(\frac{1}{8C_1},1\Big)$.

In the same manner, we easily derive (2.5.87) and (2.5.88) from (2.5.47)–(2.5.63) and theorem 2.2.2. □

Finally, we establish estimates on $\|u_{xxx}(t)\|_{H^1}$, $\|\theta_{xxx}(t)\|_{H^1}$ and $\|Z_{xxx}(t)\|_{H^1}$ based on lemmas 5.5.4 and 5.5.5.

Lemma 2.5.6. *Under assumptions of theorem 2.2.3, for any $(u_0,v_0,\theta_0,Z_0)\in\mathcal{H}^4$, there exists a positive constant $\gamma_4\le\gamma_4^{(2)}$ such that for any fixed $\gamma\in(0,\gamma_4]$, the following estimates hold for any $t>0$,*

$$\begin{aligned}&e^{\gamma t}\Big(\|u_{tt}(t)\|^2+\|u_{xt}(t)\|^2+\|\theta_{tt}(t)\|^2+\|\theta_{xt}(t)\|^2+\|Z_{tt}(t)\|^2+\|Z_{xt}(t)\|^2\Big)\\&\quad+\int_0^t e^{\gamma s}\Big(\|u_{txx}\|^2+\|u_{ttx}\|^2+\|\theta_{txx}\|^2+\|\theta_{ttx}\|^2+\|Z_{txx}\|^2+\|Z_{ttx}\|^2\Big)(s)\,ds\le C_4,\end{aligned} \quad (2.5.89)$$

$$e^{\gamma t}\|v_{xxx}(t)\|_{H^1}^2+\int_0^t e^{\gamma s}\|v_{xxx}(s)\|_{H^1}^2\,ds\le C_4, \quad (2.5.90)$$

$$\begin{aligned}&e^{\gamma t}\Big(\|u_{xxx}(t)\|_{H^1}^2+\|\theta_{xxx}(t)\|_{H^1}^2+\|Z_{xxx}(t)\|_{H^1}^2+\|u_{txx}(t)\|_{H^1}^2+\|\theta_{txx}(t)\|_{H^1}^2\\&\quad+\|Z_{txx}(t)\|_{H^1}^2\Big)+\int_0^t e^{\gamma s}(\|u_{xxxx}\|_{H^1}^2+\|\theta_{xxxx}\|_{H^1}^2+\|Z_{xxxx}\|_{H^1}^2+\|u_{txx}\|_{H^1}^2\\&\quad+\|\theta_{txx}\|_{H^1}^2+\|Z_{txx}\|_{H^1}^2+\|u_{tt}\|^2+\|\theta_{tt}\|^2+\|Z_{tt}\|^2)(s)\,ds\le C_4.\end{aligned} \quad (2.5.91)$$

Proof. Multiplying (2.5.83), (2.5.84) and (2.5.85) by ε, $\varepsilon^{\frac{3}{2}}$ and ε, respectively, then adding up the resultants and (2.5.86)–(2.5.88), and picking $\varepsilon > 0$ small enough, we can obtain the estimate (2.5.89).

Multiplying (2.5.62) by $e^{\gamma t}$, using (2.5.60), (2.5.89) and theorem 2.2.1 and choosing $\gamma > 0$ so small that $0 < \gamma \leq \gamma_4 = \min\left(\frac{1}{2C_1}, \gamma_4^{(2)}\right)$, we conclude that for any $t > 0$,

$$e^{\gamma t}\left\|\frac{v_{xxx}}{v}(t)\right\|^2 + \frac{1}{2C_1}\int_0^t e^{\gamma s}\left\|\frac{v_{xxx}}{v}(s)\right\|^2 ds \leq C_4 + C_1\int_0^t e^{\gamma s}\|E_1(s)\|^2 ds \leq C_4$$

whence

$$e^{\gamma t}\|v_{xxx}(t)\|^2 + \int_0^t e^{\gamma s}\|v_{xxx}(s)\|^2 ds \leq C_4 \tag{2.5.92}$$

which, along with (2.5.68), (2.5.69), (2.5.72), (2.5.73), (2.5.77), (2.5.78), (2.5.89), similarly to (2.5.64), leads to that for any fixed $\gamma \in (0, \gamma_4]$,

$$\begin{aligned} &e^{\gamma t}(\|u_{xxx}(t)\|_{H^1}^2 + \|\theta_{xxx}(t)\|_{H^1}^2 + \|Z_{xxx}(t)\|_{H^1}^2 + \|u_{txx}(t)\|_{H^1}^2 + \|\theta_{txx}(t)\|_{H^1}^2 \\ &\quad + \|Z_{txx}(t)_{H^1}^2\|) + \int_0^t e^{\gamma s}(\|u_{xxx}\|_{H^1}^2 + \|\theta_{xxx}\|_{H^1}^2 + \|Z_{xxx}\|_{H^1}^2 + \|u_{txx}\|_{H^1}^2 \\ &\quad + \|\theta_{txx}\|_{H^1}^2 + \|Z_{txx}\|_{H^1}^2)(s)\,ds \leq C_4 \end{aligned} \tag{2.5.93}$$

and

$$\int_0^t e^{\gamma s}\Big(\|u_{tt}\|^2 + \|\theta_{tt}\|^2 + \|Z_{tt}\|^2 + \|u_{txxx}\|^2 + \|\theta_{txxx}\|^2 + \|Z_{txxx}\|^2\Big)(s)\,ds \leq C_4. \tag{2.5.94}$$

Similarly to (2.5.92), multiplying (2.5.75) by $e^{\gamma t}$, using (2.5.71), (2.5.89), (2.5.92)–(2.5.94) and theorem 2.2.1, we conclude that for any fixed $\gamma \in (0, \gamma_4]$ and any $t > 0$,

$$e^{\gamma t}\left\|\frac{v_{xxxx}}{v}(s)\right\|^2 + \frac{1}{2C_1}\int_0^t e^{\gamma s}\left\|\frac{v_{xxxx}}{v}(s)\right\|^2 ds \leq C_4 + C_1\int_0^t e^{\gamma s}\|E_2(s)\|^2 ds \leq C_4 \tag{2.5.95}$$

whence

$$e^{\gamma t}\|v_{xxxx}(t)\|^2 + \int_0^t e^{\gamma s}\|v_{xxxx}(s)\|^2 ds \leq C_4. \tag{2.5.96}$$

Similarly to (2.5.79)–(2.5.82), we derive that for any fixed $\gamma \in (0, \gamma_4]$,

$$\int_0^t e^{\gamma s}\Big(\|u_{xxxxx}\|^2 + \|\theta_{xxxxx}\|^2 + \|Z_{xxxxx}\|^2\Big)(s)\,ds \leq C_4. \tag{2.5.97}$$

Finally, we derive the desired estimates (2.5.90) and (2.5.91) from (2.5.89) and (2.5.92)–(2.5.96). □

Till now we have completed the proof of theorem 2.2.3. □

2.6 Bibliographic Comments

Combustion is, to a large extent, a superset of fluid mechanics, but – unlike that subject – one does not find within it a rich interaction of mathematics and experiment. In recent years, applied mathematicians have been attracted in growing numbers by a subject whose richness affords ample scope for their talents. There are some related works concerning the combustion of the compressible viscous radiative and reactive gas. Donatelli and Trivisa [22] studied a multi-dimensional model for the dynamic combustion of a viscous, compressible, radiative and reactive gas with higher order kinetics, and the pressure depends not only on the density and temperature but also on the mass fraction of the reactant. By the spirit of Feireisl [39], Feireisl and Novotný [40] established the existence of global weak solutions using weak convergence methods, compactness and interpolation arguments and extended the results of Donatelli and Trivisa [23], where the pressure depended only on the density and temperature. For the three-dimensional case, we can also see the results in Ducomet and Feireisl [29, 30] and Feireisl *et al.* [41]. For the one-dimensional case with various boundary conditions, the global well-posedness of cylindrically symmetric solutions have been studied by many researchers. We can refer to [13, 15–17, 26, 47, 71, 108, 112, 113, 129, 137] and the references therein.

In recent years, compressible Navier–Stokes systems with additional correction terms in the pressure, the internal energy and the thermal conduction modeling the radiative hydrodynamics have been paid a lot of attention (see [26–30, 34–36, 108, 109, 112, 142]). Ducomet [27] established the global existence and exponential decay in H^1 of solutions to the one-dimensional model for $q \geq 4$ with the Dirichlet–Neumann boundary conditions. Later on, Ducomet and Zlotnik [34–36] established the existence of global solutions to the one-dimensional model under rather general assumptions on q and κ. Moreover, they obtained the exponential stabilization for solutions by constructing new Lyapunov functionals. Recently, Zhang and Xie [142] proved the global well-posedness of the one-dimensional MHD with thermal radiation and self-gravitation effects for the Dirichlet–Neumann boundary value problem under quite general assumptions on κ and for $q > \frac{5}{2}$. Qin and Hu [108] improved Zhang and Xie's results for $q > \frac{2+\sqrt{211}}{9} \approx 1.83$. On the subject of global existence and large time behavior of smooth/strong solutions for the one-dimensional motions of viscous polytropic ideal gas under various conditions, we refer the readers to Kazhikhov and Shelukhin [72], Kawohl [71], Chen [13], Jiang [62–64], Zheng and Qin [144], Qin [99, 103, 104], etc. For free boundary problems, we would like to mention the work of Nagasawa [89, 90], Tani [129, 130] and Hsiao and Luo [56].

For the free and pure Neumann boundary value problem, Umehara and Tani [134] established the global existence, uniqueness of a classical solutions under assumptions $4 \leq q \leq 16$ and $0 \leq \beta \leq \frac{13}{2}$. Soon after, Umehara and Tani [135] improved the range of (q, β) to $q \geq 3$ and $0 \leq \beta < q + 9$ which includes the most interesting physical case $q = 3$. Later on, Qin and Huang [110] gave a refined proof of the global existence and exponential stability with the smallness assumption imposed on initial datum v_0, where $(q, \beta) \in E$ with $E = E_1 \cup E_2$ and

$$E_1 = \left\{ (q, \beta) \in \mathbb{R}^2 : \frac{12}{5} < q, \ \ 0 < \beta \leq 8 \right\}, \tag{2.6.1}$$

$$\begin{aligned} E_2 = &\left\{ (q, \beta) \in \mathbb{R}^2 : \frac{12}{5} < q \leq 3, \ \ \beta < 2q + 6, \ \ \beta > 8 \right\} \\ &\cup \{ (q, \beta) \in \mathbb{R}^2 : 3 \leq q, \ \ \beta < q + 9, \ \ \beta > 8 \}. \end{aligned} \tag{2.6.2}$$

Subsequently, Qin *et al.* [112] further strengthened the results for $(q, \beta) \in E$, where $E = E_1 \cup E_2$ and

$$E_1 = \left\{ (q, \beta) \in \mathbb{R}^2 : 2 \leq q < 3, \ \ 0 \leq \beta < 2q + 6 \right\}, \tag{2.6.3}$$

$$E_2 = \left\{ (q, \beta) \in \mathbb{R}^2 : 3 \leq q, \ \ 0 \leq \beta < q + 9 \right\}. \tag{2.6.4}$$

Recently, Jiang and Zheng [68] obtained existence results for $q \geq 2$, $0 \leq \beta < q + 9$. Jiang and Zheng [69] obtained similar existence results of the one-dimensional viscous radiative and reactive gas with the higher order kinetics under the same assumptions. Moreover, the asymptotic behavior and exponential stability of global solutions can also be stated, but there are assumptions of smooth initial data $(v_0, u_0, \theta_0, Z_0) \in C^{1+\alpha}(\boldsymbol{\Omega}) \times (C^{2+\alpha}(\boldsymbol{\Omega}))^3$ in [68] and $(v_0, u_0, \theta_0, Z_0) \in H^1(\boldsymbol{\Omega}) \times H^2(\boldsymbol{\Omega}) \times H^2(\boldsymbol{\Omega}) \times H^2(\boldsymbol{\Omega})$ in [69]. Especially, in [68, 69], they imposed the following assumptions on the heat conductivity $\kappa = \kappa(v, \theta)$:

(F1) κ is strictly positive, *i.e.*, there exists a positive constant $\underline{\kappa}$ such that $0 < \underline{\kappa} \leq \kappa(v, \theta)$.
(F2) $\kappa(\cdot, \cdot) \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover, $\kappa(v, \theta)$ satisfies the following growth condition for some $q \geq 2$:

$$\begin{aligned} &\text{for any } \delta > 1, \ \forall v \in \left[\delta^{-1}, \delta\right], \theta > 0, \\ &0 < \kappa_1(\delta)(1 + \theta)^q \leq \kappa(v, \theta), \\ &(\kappa + |\kappa_v| + |\kappa_\theta| + |\kappa_{vv}|)(v, \theta) \leq \kappa_2(\delta)(1 + \theta)^q. \end{aligned}$$

(F3) For any $\delta > 0$, there exists a constant $\kappa_0(\delta) > 0$ and some $r \in [2, \min(8, q)]$ such that for $v \geq \delta$,

$$\kappa(v, \theta)/v \geq \kappa_0(\delta)\theta^r.$$

For spherically symmetric Navier–Stokes equations, we would like to refer to [48, 51, 64, 67, 139] and the references therein.

Chapter 3

Global Existence, Uniqueness and Exponential Stability of Solutions for the One-Dimensional Navier–Stokes Equations with Capillarity

3.1 Introduction

In this chapter, we shall study compressible fluids endowed with internal capillarity. In order to model fluid capillarity effects, the formulation of the theory of capillarity with diffuse interfaces was first introduced by Korteweg [74] a century ago, and actually derived by Dunn and Serrin [37] in the 1980s using the second gradient theory (see also [61, 132]). At first, the Young–Laplace theory claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. Here the main difficulty consists in describing the location and the movement of the interfaces.

Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface) or whether the phase boundary corresponds to a more regular transition (diffuse interface). The sharp interface models can be regarded as a problem with free boundary, while the diffuse interface models have the advantage of considering only one set of equations in a single spatial domain (the density takes into account the different phases), which remarkably simplifies the mathematical and numerical studies. An interesting use of this model is the application to liquid–vapour mixtures with phase changes.

Korteweg-type models are based on an extended version of non-equilibrium thermodynamics, which supposes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. It is a hot topic on the study of this model in the field of fluid mechanics and applied mathematics, especially partial differential equations.

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3.2 Model and Main Results

3.2.1 Korteweg-Type Model

Let us consider a fluid of density $\rho \geq 0$, velocity field $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, energy density e and temperature θ.

The following compressible capillary fluid model can be derived from a Cahn–Hilliard-like free energy (see the pioneering work by Dunn and Serrin in [37] and also [2, 9, 49, 60]), and we take the equations of balance of mass, momentum and energy as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = \mathrm{div}(S + K), \\ \partial_t\left(\rho\left(e + \dfrac{v^2}{2}\right)\right) + \mathrm{div}\left(\rho v\left(e + \dfrac{v^2}{2}\right)\right) = \mathrm{div}(\alpha \nabla \theta) + \mathrm{div}((S+K)\cdot v), \end{cases}$$

where the viscous stress tensor $S = (S_{i,j})$ and the Korteweg stress tensor $K = (K_{i,j})$ read as (see [2, 9, 37, 49, 60])

$$\begin{cases} S_{i,j} = (\lambda \mathrm{div} v - P(\rho, \theta))\delta_{i,j} + 2\mu d_{i,j}(v), \\ K_{i,j} = \dfrac{\kappa}{2}(\Delta \rho^2 - |\nabla \rho|^2)\delta_{i,j} - \kappa \partial_i \rho \partial_j \rho, \end{cases}$$

where $d_{i,j}(v) = \frac{\partial_i v_j + \partial_j v_i}{2}$ is the strain tensor, and the Lamé viscosity coefficients λ and μ depend on the density ρ and satisfy $\mu > 0$ and Stokes' law $n\lambda + 2\mu > 0$. The thermal conduction coefficient $\alpha = \alpha(\rho,\ \theta)$ is a given nonnegative function of temperature θ and density ρ, coefficient of capillarity κ is a regular function. Specially, notice that when $\kappa = 0$, the system reduces to the compressible Navier–Stokes system. The $\mathrm{div} K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid–vapour. P is a general pressure term.

In this chapter, we are interested in the following one-dimensional non-isothermal Navier–Stokes equations with capillarity, which comes from Slemrod [125] and Dunn and Serrin [37],

$$\begin{cases} u_t = v_x, & (3.2.1) \\ v_t = \sigma_x, & (3.2.2) \\ \left(e + \dfrac{v^2}{2} + C(u)\dfrac{u_x^2}{2}\right)_t = (\sigma v)_x + Q_x + (C(u)u_x v_x)_x, & (3.2.3) \end{cases}$$

where subscripts indicate partial differentiations, u, v, σ, e and Q stand for the specific volume, velocity, stress, internal energy and heat flux, respectively. The stress σ is given by Korteweg's theory of capillarity, *i.e.*,

$$\sigma = -P(u,\theta) + \mu(u,\theta)\frac{v_x}{u} - \frac{1}{2}C'(u)u_x^2 - C(u)u_{xx}, \tag{3.2.4}$$

where $C(u)$ is the capillarity coefficient, $\mu(u, \theta)$ is the viscosity coefficient, P is the pressure, θ is the absolute temperature. Note that u, θ and e may take only positive values. We take the heat flux Q satisfying Fourier's law

$$Q(u,\theta,\theta_x) = \alpha(u,\theta)\frac{\theta_x}{u}, \tag{3.2.5}$$

and let the capillarity $C(u) = \kappa$ be a positive constant and hence the Korteweg tensor $K = K(u_{xx}) = -\kappa u_{xx}$ in order to simplify calculations. Thus, we wish to prove the global existence, uniqueness and exponential stability of solutions for the following system with the reference density $\rho_0 = 1$

$$\begin{cases} u_t = v_x, & (3.2.6)\\ v_t = -P_x + \left(\mu\dfrac{v_x}{u}\right)_x - \kappa u_{xxx}, & (3.2.7)\\ \left(e + \dfrac{v^2}{2} + \kappa\dfrac{u_x^2}{2}\right)_t = \left(-Pv + \mu\dfrac{v_x}{u}v - \kappa u_{xx}v\right)_x + \left(\alpha(u,\theta)\dfrac{\theta_x}{u}\right)_x + \kappa(u_x v_x)_x, & (3.2.8)\end{cases}$$

in the region $\{0 \le x \le 1,\ t \ge 0\}$ under initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in [0,1] \tag{3.2.9}$$

and the boundary conditions, namely, homogeneous Dirichlet conditions for the velocity

$$v(0,t) = v(1,t) = 0, \tag{3.2.10}$$

and the Neumann conditions for the specific volume

$$u_x(0,t) = u_x(1,t) = 0, \tag{3.2.11}$$

and the temperature

$$\theta_x(0,t) = \theta_x(1,t) = 0. \tag{3.2.12}$$

The second law of thermodynamics is expressed by the Clausius–Duhem inequality

$$\eta_t + \left(\frac{Q}{\theta}\right) \ge 0. \tag{3.2.13}$$

For a one-dimensional homogeneous real gas, e, σ, η and Q are given by the constitutive relations

$$e = e(u,\theta), \ \ \sigma = \sigma(u,\theta,\theta_x), \ \ \eta = \eta(u,\theta), \ \ Q = Q(u,\theta,\theta_x) \tag{3.2.14}$$

which in order to be consistent with (3.2.13) must satisfy

$$\begin{aligned}\sigma(u,\theta,0) = \mathbf{\Psi}_u(u,\theta),\ \eta(u,\theta)\\ = -\mathbf{\Psi}_\theta(u,\theta),\ (\sigma(u,\theta,w) - \sigma(u,\theta,0))w \ge 0,\ Q(u,\theta,g)g \le 0,\end{aligned} \tag{3.2.15}$$

$$e_u(u,\theta) = -P(u,\theta) + \theta P_\theta(u,\theta), \tag{3.2.16}$$

$$P_\theta(u,\theta) = \eta_u(u,\theta),\ e_\theta(u,\theta) = \theta\eta_\theta(u,\theta) = -\theta\Psi_{\theta\theta}(u,\theta), \tag{3.2.17}$$

where $\Psi(u,\theta) = e(u,\theta) - \theta\eta(u,\theta)$ is the Helmholtz free energy function.

Compared with the case of compressible Navier–Stokes system [55, 58, 59], the main difficulty we encounter here is the appearance of the higher order of the specific volume u which brings more difficulties to derive the uniform *a priori* estimates for solutions of the initial boundary value problems.

We assume that e, P, σ and α are smooth sufficiently (to be precise later on in theorems 3.2.1–3.2.3) on $0 < u < +\infty$ and $0 \leq \theta < +\infty$, and there exist exponents q and r satisfying one of the following relations

$$\begin{cases} \dfrac{1}{5} < r \leq \dfrac{1}{3}, \quad \dfrac{1-r}{2} < q, & (3.2.18)\\ \dfrac{1}{3} < r \leq \dfrac{3}{2}, \quad \dfrac{4r+1}{7} < q, & (3.2.19)\\ \dfrac{3}{2} < r, \quad \dfrac{2r+1}{4} < q. & (3.2.20) \end{cases}$$

In order to describe clearly growth conditions with respect to the temperature, we require that there be positive exponents r and positive constants ϵ, p_1, p_2, α_0, and that for any $\underline{u} > 0$, there be positive constants $N(\underline{u})$, $p_3(\underline{u})$, $p_4(\underline{u})$ and $\alpha_1(\underline{u})$ such that for all $u \geq \underline{u}$, $w \in \mathbb{R}$ and $\theta \geq 0$, the following conditions hold

$$e(u,0) \geq 0,\ \epsilon(1+\theta^r) \leq e_\theta(u,\theta) \leq N(\underline{u})(1+\theta^r), \tag{3.2.21}$$

$$0 < p_1 \leq u[P(u,\theta) - K(w)] \leq p_2(1+\theta^{1+r}), \tag{3.2.22}$$

$$-p_3(\underline{u})(1+\theta^{1+r}) \leq P_u(u,\theta) \leq -p_4(\underline{u})(1+\theta^{1+r}), \tag{3.2.23}$$

$$|P_\theta(u,\theta)| \leq p_4(\underline{u})(1+\theta^r), \tag{3.2.24}$$

$$\alpha_0(1+\theta^q) \leq \alpha(u,\theta) \leq \alpha_1(\underline{u})(1+\theta^q), \tag{3.2.25}$$

$$|\alpha_u(u,\theta)| + |\alpha_{uu}(u,\theta)| \leq \alpha_1(\underline{u})(1+\theta^q). \tag{3.2.26}$$

The above growth conditions are motivated by the facts in [3, 140], where it is pointed out that e grows as θ^{1+r} with $r \approx 0.5$ and α increases like θ^q with $4.5 \leq q \leq 5.5$. In addition, we suppose that the viscosity $\mu(u,\theta)$ is independent of θ, uniformly positive, and bounded, *i.e.*,

$$\mu(u,\theta) = \mu_0 > 0. \tag{3.2.27}$$

In order to establish the existence and exponential stability of time globally defined solutions to problem (3.2.6)–(3.2.12), we encounter the main difficulty from the higher order nonlinearities of θ and partial derivatives of u in system (3.2.6)–(3.2.8). To overcome them, we make full use of corollaries 3.3.1 and 3.3.2 to reduce the higher order of θ and interpolation technique, and our main idea comes from Qin [99, 103, 104].

In order to state our results clearly, we define the following three classes of functions as

$$\begin{aligned} H_+^1 = \{(u, v, \theta) \in H^2[0,1] \times H^1[0,1] \times H^1[0,1] : u(x) > 0, \theta(x) > 0, x \in [0,1],\\ u'(0) = u'(1) = 0,\ v(0) = v(1) = 0\}, \end{aligned}$$

$$\begin{aligned} H_+^2 = \{(u, v, \theta) \in H^3[0,1] \times H^2[0,1] \times H^2[0,1] : u(x) > 0, \theta(x) > 0, x \in [0,1],\\ u'(0) = u'(1) = 0,\ v(0) = v(1) = 0,\ \theta'(0) = \theta'(1) = 0\}, \end{aligned}$$

and

$$\begin{aligned} H_+^4 = \{(u, v, \theta) \in H^5[0,1] \times H^4[0,1] \times H^4[0,1] : u(x) > 0, \theta(x) > 0, x \in [0,1],\\ u'(0) = u'(1) = 0,\ v(0) = v(1) = 0,\ \theta'(0) = \theta'(1) = 0\}, \end{aligned}$$

which become the metric spaces when equipped with the metrics induced from the usual norms.

In general, without the danger of confusion, we will use the same symbol to denote state functions as well as their values along a thermodynamic process, *e.g.*, $P(u, \theta)$ and $P(u(x, t), \theta(x, t))$. Letters C_i $(i = 1, 2, 4)$ will denote the universal constants depending on the norms of initial datum (u_0, v_0, θ_0) in H_+^i, $\min_{x\in[0,1]} u_0(x)$ and $\min_{x\in[0,1]} \theta_0(x)$, but being independent of t, respectively.

3.2.2 Main Theorems

We are now in a position to state our main theorems.

Theorem 3.2.1. *Assume that e, P, σ and α are C^2 functions on $u \in (0, +\infty)$ and $\theta \in [0, +\infty)$, and assumptions (3.2.13)–(3.2.27) hold. Then the problem (3.2.6)–(3.2.12) admits the unique generalized global solution $(u(x, t), v(x, t), \theta(x, t))$ in H_+^1, which defines a nonlinear C_0-semigroup $S(t)$ on H_+^1. Moreover, for any $(u_0, v_0, \theta_0) \in H_+^1$, there exists a constant $\gamma_1 = \gamma_1(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_1]$ and for any $t \in (0, +\infty)$, the following estimate holds:*

$$\begin{aligned} \left\|(u(t), v(t), \theta(t)) - (\overline{u}, 0, \overline{\theta})\right\|_{H_+^1}^2 &= \left\|S(t)(u_0, v_0, \theta_0) - (\overline{u}, 0, \overline{\theta})\right\|_{H_+^1}^2 \\ &\le C_1 e^{-\gamma t}, \end{aligned} \tag{3.2.28}$$

that is, the semigroup $S(t)$ *is exponentially stable on* H_+^1*. Here*

$$\overline{u} = \overline{u_0} = \int_0^1 u_0(x)\,dx \tag{3.2.29}$$

and the constant $\bar{\theta} > 0$ *satisfies*

$$e(\overline{u}, \bar{\theta}) = \int_0^1 \left(e(u_0, \theta_0) + \frac{v_0^2}{2} \right)(x)\,dx \tag{3.2.30}$$

and $(\overline{u_0}, 0, \bar{\theta})$ *is the unique solution to the corresponding stationary problem to (3.2.6)–(3.2.12).*

Theorem 3.2.2. *Assume that e, P, σ and α are* C^3 *functions on* $u \in (0, +\infty)$ *and* $\theta \in [0, +\infty)$*, and assumptions (3.2.13)–(3.2.27) hold. Then the problem (3.2.6)–(3.2.12) admits the unique generalized global solution* $(u(x,t), v(x,t), \theta(x,t))$ *in* H_+^2*, which defines a nonlinear* C_0*-semigroup* $S(t)$ *on* H_+^2*. Moreover, for any* $(u_0, v_0, \theta_0) \in H_+^2$*, there exists a constant* $\gamma_2 = \gamma_2(C_2) > 0$ *such that for any fixed* $\gamma \in (0, \gamma_2]$ *and for any* $t \in (0, +\infty)$*, the following estimate holds:*

$$\begin{aligned} \left\| (u(t), v(t), \theta(t)) - (\overline{u}, 0, \overline{\theta}) \right\|_{H_+^2}^2 &= \left\| S(t)(u_0, v_0, \theta_0) - (\overline{u}, 0, \overline{\theta}) \right\|_{H_+^2}^2 \\ &\le C_2 e^{-\gamma t}, \end{aligned} \tag{3.2.31}$$

that is, the semigroup $S(t)$ *is exponentially stable on* H_+^2*. Here* $\overline{u}$ *and* $\bar{\theta}$ *are the same as in (3.2.29), (3.2.30) in theorem 3.2.1.*

Theorem 3.2.3. *Assume that e, P, σ and α are* C^5 *functions on* $u \in (0, +\infty)$ *and* $\theta \in [0, +\infty)$*, and assumptions (3.2.13)–(3.2.27) hold. Then the problem (3.2.6)–(3.2.12) admits the unique generalized global solution* $(u(x,t), v(x,t), \theta(x,t))$ *in* H_+^4*, which also defines a nonlinear* C_0*-semigroup* $S(t)$ *on* H_+^4*. Moreover, for any* $(u_0, v_0, \theta_0) \in H_+^4$*, there exists a constant* $\gamma_4 = \gamma_4(C_4) > 0$ *such that for any fixed* $\gamma \in (0, \gamma_4]$ *and for any* $t \in (0, +\infty)$*, the following estimate holds:*

$$\begin{aligned} \left\| (u(t), v(t), \theta(t)) - (\overline{u}, 0, \overline{\theta}) \right\|_{H_+^4}^2 &= \left\| S(t)(u_0, v_0, \theta_0) - (\overline{u}, 0, \overline{\theta}) \right\|_{H_+^4}^2 \\ &\le C_4 e^{-\gamma t}, \end{aligned} \tag{3.2.32}$$

that is, the semigroup $S(t)$ *is exponentially stable on* H_+^4*. Here* $\overline{u}$ *and* $\bar{\theta}$ *are the same as in (3.2.29), (3.2.30) in theorem 3.2.1.*

Remark 3.2.1. *Here we consider the non-isothermal case and our model is the more generalized than those in [127, 133].*

Remark 3.2.2. *We have not obtained similar results with* $0 \le r \le \frac{1}{5}$ *in this work, so we may study this case by the other technique in the future.*

The results of this chapter are selected from [121]. Our main novelties are as follows. (i) Slemrod [122] only considered the thermal conductivity $\alpha(u,\theta)$ is positive constant, but here we consider the thermal conductivity $\alpha(u,\theta)$ depends not only on the specific volume but also on the absolute temperature. (ii) In the proof of existence and exponential stability of global strong solutions, we make full use of the higher order partial derivatives of the specific volume. Moreover, our method and initial value space are different from those in Kotschote [76] who only proved the existence and uniqueness of strong solutions when the initial data $(u_0, v_0, \theta_0) \in W^{3-2/p,p} \times W^{2-2/p,p} \times W^{2-2/p,p}$ $(n+2 < p < +\infty,\ n \geq 1)$.

3.3 Global Existence and Exponential Stability in H^1_+

In this section, we shall obtain the global existence, uniqueness, asymptotic behavior and exponential stability of solutions to the problem (3.2.6)–(3.2.12), that is, we shall complete the proof of theorem 3.2.1 in terms of a series of lemmas.

3.3.1 Global Existence in H^1_+

First, we establish some basic inequalities or equalities by the following two lemmas.

Lemma 3.3.1. *Under assumptions of theorem 3.2.1, we have the following estimates,*

$$\theta(x,t) > 0 \quad on\,[0,1]\times[0,+\infty), \tag{3.3.1}$$

$$\int_0^1 \left(e + \frac{v^2}{2} + \kappa\frac{u_x^2}{2}\right)dx = \int_0^1 \left(e(x,0) + \frac{v_0^2}{2}(x) + \kappa\frac{u_{0x}^2}{2}(x)\right)dx = E_0,\ \forall t > 0, \tag{3.3.2}$$

$$\int_0^1 (\theta + \theta^{1+r})(x,t)\,dx \leq C,\ \forall t > 0, \tag{3.3.3}$$

$$\int_0^1 u(x,t)\,dx = \int_0^1 u_0(x)\,dx = \overline{u_0},\ \forall t > 0. \tag{3.3.4}$$

Proof. Using equations (3.2.6) and (3.2.7), we can obtain from (3.2.8)

$$e_\theta(u,\theta)\theta_t + \theta P_\theta v_x - \mu_0\frac{v_x^2}{u} = \left(\alpha(u,\theta)\frac{\theta_x}{u}\right)_x. \tag{3.3.5}$$

Thus estimates (3.3.1), (3.3.3) and (3.3.4) can be proved in a similar way to that in lemma 2.3.1 (see also Qin [99]). Integrating (3.2.6) over $Q_t := [0,1]\times[0,t]$, we can also get (3.3.2), the conservation law of total energy. □

Lemma 3.3.2. *Under assumptions of theorem 3.2.1, the following estimate holds that for any* $t \in (0, +\infty)$,

$$\int_0^1 \left(\theta - \ln\theta - 1 + \theta^{1+r} + \frac{v^2}{2} + \kappa\frac{u_x^2}{2}\right) dx + \int_0^t \int_0^1 \left(\frac{v_x^2}{u\theta} + \frac{(1+\theta^q)\theta_x^2}{u\theta^2}\right) dxds \le C_1. \tag{3.3.6}$$

Proof. Denote

$$E(u,\theta) := \Psi(u,\theta) - \Psi(1,1) - \Psi_u(1,1)(u-1) - \Psi_\theta(u,\theta)(\theta-1). \tag{3.3.7}$$

By (3.2.9) and (3.2.10), we derive from (3.3.5) and (3.2.14)–(3.2.17) after a direct calculation that

$$\begin{aligned}&\left(E(u,\theta) + \frac{v^2}{2} + \kappa\frac{u_x^2}{2}\right)_t + \mu_0\frac{v_x^2}{u\theta} + \frac{\alpha(u,\theta)\theta_x^2}{u\theta^2}\\ &\quad = \left(\sigma v + \frac{(\theta-1)\alpha(u,\theta)\theta_x}{u\theta}\right)_x - \Psi_u(1,1)v_x + \kappa(u_x v_x)_x.\end{aligned} \tag{3.3.8}$$

Integrating (3.3.8) over Q_t and using (3.2.10)–(3.2.12), we obtain

$$\begin{aligned}&\int_0^1 \left(E(u,\theta) + \frac{v^2}{2} + \kappa\frac{u_x^2}{2}\right)(x,t)\,dx + \int_0^t\int_0^1 \left(\mu_0\frac{v_x^2}{u\theta} + \frac{\alpha(u,\theta)\theta_x^2}{u\theta^2}\right) dxds\\ &\quad = \int_0^1 \left(E(u_0,\theta_0) + \frac{v_0^2}{2} + \kappa\frac{u_{0x}^2}{2}\right) dx.\end{aligned} \tag{3.3.9}$$

In view of (3.2.23), we have $\Psi_{uu}(u,\,1) = -P(u,\,1) > 0$ for all $u > 0$. Therefore it follows from the Taylor theorem and (3.2.21) that

$$\begin{aligned}&E(u,\theta) - \Psi(u,\theta) + \Psi(u,1) + (\theta-1)\Psi_\theta(u,\theta)\\ &\quad = \Psi(u,1) - \Psi(1,1) - \Psi_u(1,1)(u-1)\\ &\quad = (u-1)^2 \int_0^1 (1-\xi)\Psi_{uu}(1+\xi(u-1),1)\,d\xi \ge 0.\end{aligned}$$

Thus,

$$\begin{aligned}E(u,\theta) &\ge \Psi(u,\theta) - \Psi(u,1) - (\theta-1)\Psi_\theta(u,\theta)\\ &\ge -(1-\theta)^2 \int_0^1 (1-\tau)\Psi_{\theta\theta}(u,\theta+\tau(1-\theta))\,d\tau\\ &\ge (1-\theta)^2 \int_0^1 (1-\tau)\frac{e_\theta(u,\theta+\tau(1-\theta))}{\theta+\tau(1-\theta)}\,d\tau\\ &\ge v(1-\theta)^2 \int_0^1 \frac{(1-\tau)(1+(\theta+\tau(1-\theta))^r)}{\theta+\tau(1-\theta)} d\tau,\end{aligned}$$

i.e.,

$$E(u,\theta) \geq \begin{cases} v(\theta - \ln\theta - 1) + \dfrac{v(1-\theta^r)}{r} - \dfrac{v(1-\theta^{1+r})}{r+1}, & \text{for } r > 0 \\ 2v(\theta - \ln\theta - 1), & \text{for } r = 0, \end{cases}$$

which, together with (3.2.23) and (3.3.9), leads to (3.3.6). □

The next two lemmas give us two expressions of the specific volume $u(x, t)$ which play an important role in our analysis.

Lemma 3.3.3. *For any $t \in [0, +\infty)$, there exists one point $x_1 = x_1(t) \in [0, 1]$ such that the solution $u(x, t)$ to problem (3.2.6)–(3.2.12) can be expressed as*

$$u(x,t) = D(x,t)Z(t)\left(1 + \mu_0^{-1}\int_0^t \frac{u(x,s)(P-K)(x,s)}{D(x,s)Z(s)}ds\right), \tag{3.3.10}$$

where

$$\begin{aligned} D(x,t) = u_0(x)\exp\Bigg\{\mu_0^{-1}\Bigg(&\int_{x_1(t)}^x v(y,t)\,dy - \int_0^x v_0(y)\,dy \\ &+ \frac{1}{\overline{u_0}}\int_0^1 u_0(x)\int_0^x v_0(y)\,dydx\Bigg)\Bigg\}, \end{aligned} \tag{3.3.11}$$

$$Z(t) = \exp\left[-\frac{1}{\mu_0\overline{u_0}}\int_0^t\int_0^1 (v^2 + u(P-K))(y,s)\,dyds\right]. \tag{3.3.12}$$

Proof. Replacing $P - \delta$ by $P - K$ and letting $\delta = 0$, then going along the line of argumentation in the proof of lemma 2.3.2, we can complete the proof of this lemma. See also lemma 2.1.3 in Qin [99]. □

Lemma 3.3.4. *For any $t \in [0, +\infty)$, there exists one point $a(t) \in [0,1]$ such that the solution $u(x, t)$ to the problem (3.2.6)–(3.2.12) can also be expressed as*

$$u(x,t) = B^{-1}(x,t)\,Y^{-1}(t)\left[1 + \frac{1}{\mu_0}\int_0^t u(x,s)(P-K)(x,s)\,Y(s)B(x,s)\,ds\right] \tag{3.3.13}$$

where

$$Y(t) = u_0(a(t))\exp\left(\frac{1}{\mu_0}\int_0^t (P-K)(x,s)\,ds\right), \tag{3.3.14}$$

$$B(x,t) = \frac{1}{u_0(x)\overline{u_0}}\exp\left(\frac{1}{\mu_0}\int_{a(t)}^x (v_0(y) - v(y,t))\,dy\right), \tag{3.3.15}$$

$$u(a(t),t) = \overline{u_0}. \tag{3.3.16}$$

Proof. Replacing $P-\delta$ by $P-K$ and letting $\delta=0$, then repeating the same argumentation in the proof of lemma 2.3.2, we can also prove this lemma. See also lemma 2.1.4 in Qin [99]. □

Based on the above lemma, we can obtain the next lemma on estimates for uniform upper and lower bounds of the specific volume.

Lemma 3.3.5. *Under assumptions of theorem 3.2.1, the solution $u(x,\ t)$ to the problem (3.2.6)–(3.2.12) has the upper and positive lower bounds, that is, for any $(x,\ t)\in[0,\ 1]\times[0,\ +\infty)$,*

$$0<C^{-1}\le u(x,t)\le C. \tag{3.3.17}$$

Proof. The proof is similar to that of lemma 2.3.3, here we omit the detail. See also lemma 2.1.5 in Qin [99]. □

Remark 3.3.1. *This lemma gives the estimate on the upper and lower bounds of specific volume u under the assumption (3.2.22). If we replace (3.2.22) with $0<p_1\le uP(u,\theta)\le p_2(1+\theta^{1+r})$, we can obtain the same result by some strict limitations on the initial energy $E_0=\int_0^1\left(e_0+\frac{1}{2}v_0^2+\frac{\kappa}{2}u_{0x}^2\right)dx\ll 1$, i.e., E_0 is chosen small enough.*

The following corollary is very important for our further analysis.

Corollary 3.3.1. *Under assumptions of theorem 3.2.1, the following estimate holds that for any $(x,t)\in[0,1]\times[0,+\infty)$,*

$$C^{-1}-CV_1(t)\le\theta^{2m_1}(x,t)\le C+CV_1(t) \tag{3.3.18}$$

with $0\le m_1\le m=(q+r+1)/2$ and $V_1(t)=\int_0^1\frac{(1+\theta^q)\theta_x^2}{\theta^2}dx$ verifying $\int_0^{+\infty}V_1(t)dt<+\infty$.

Now we turn to estimate the first-order derivative of the specific volume.

Lemma 3.3.6. *Under assumptions of theorem 3.2.1, the following estimates hold for any $t\in(0,\ +\infty)$,*

$$\int_0^t\|v(s)\|_{L^\infty}^2ds\le C_1, \tag{3.3.19}$$

$$\int_0^t\int_0^1(1+\theta)^{2m}v^2(x,s)dxds\le C_1, \tag{3.3.20}$$

$$\|u_x(t)\|^2+\int_0^t\int_0^1\left[(1+\theta^{r+1})u_x^2+u_{xx}^2\right](x,s)dxds$$
$$\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_2} \tag{3.3.21}$$

with $\beta_1=\max(r+1-q,0),\beta_2=\max(1,\beta_1)$.

Proof. Estimates (3.3.19) and (3.3.20) can be obtained by the weighted Hölder inequality, the equation (3.3.5), lemma 3.3.1 and corollary 3.3.1.

The equation (3.2.7) can be rewritten as

$$\left(v-\mu_0\frac{u_x}{u}\right)_t+(P-K)_x=0, \tag{3.3.22}$$

that is,

$$\left(v-\mu_0\frac{u_x}{u}\right)_t+P_u(u,\theta)u_x-K_x(x,t)=P_\theta(u,\theta)\theta_x.$$

Multiplying (3.3.22) by $v-\mu_0\frac{u_x}{u}$ in Q_t, we have

$$\begin{aligned}&\frac{1}{2}\left\|v-\mu_0\frac{u_x}{u}\right\|^2+\int_0^t\int_0^1\frac{-\mu_0P_u(u,\theta)u_x^2}{u}dxds+\int_0^t\int_0^1\frac{\mu_0u_xK_x}{u}dxds\\&\quad=\frac{1}{2}\left\|v_0-\mu_0\frac{u_{0x}}{u_0}\right\|^2-\int_0^t\int_0^1\left(P_uu_xv+P_\theta\theta_x\left(v-\frac{\mu_0u_x}{u}\right)\right)dxds+\int_0^t\int_0^1vK_xdxds.\end{aligned}$$

Using an integration by parts and noting the boundary conditions (3.2.10) and (3.2.12), we can obtain

$$\int_0^t\int_0^1\frac{\mu_0u_xK_x}{u}dxds=\mu_0\kappa\int_0^t\int_0^1\frac{u_{xx}^2}{u}dxds-\mu_0\kappa\int_0^t\int_0^1\frac{u_{xx}u_x^2}{u^2}dxds, \tag{3.3.23}$$

and

$$\int_0^t\int_0^1vK_xdxds=-\int_0^t\int_0^1Kv_xdxds=\kappa\int_0^t\int_0^1u_{xx}v_xdxds. \tag{3.3.24}$$

Using the Cauchy inequality and the assumption (3.2.23), we can derive from corollary 3.3.1,

$$\begin{aligned}&\left\|v-\mu_0\frac{u_x}{u}\right\|^2+\int_0^t\int_0^1\left((1+\theta^{r+1})u_x^2+u_{xx}^2\right)dxds\\&\quad\le C_1+C\int_0^t\int_0^1\left((1+\theta^{r+1})|u_xv|+(1+\theta^r)\left|\theta_x\left(v-\mu_0\frac{u_x}{u}\right)\right|+|u_{xx}u_x^2|+|u_{xx}v_x|\right)dxds\\&\quad\le C_1+C\int_0^t\int_0^1(1+\theta^{1+r})(\epsilon u_x^2+Cv^2)dxds+C\int_0^tV_1(s)ds\\&\qquad+C\int_0^t\int_0^1\frac{\theta^2(1+\theta^r)^2}{1+\theta^q}v^2dxds+C\epsilon\int_0^t\int_0^1(1+\theta^{1+r})u_x^2dxds\\&\qquad+C\int_0^t\int_0^1\frac{(1+\theta^r)^2\theta_x^2}{1+\theta^{1+r}}dxds+\epsilon\int_0^t\|u_{xx}(s)\|^2ds+C\int_0^t\|v_x(s)\|^2ds+C\int_0^t\|u_x(s)\|_{L^4}^4ds\\&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_1}+C\epsilon\int_0^t\int_0^1(1+\theta^{1+r})u_x^2dxds+C\int_0^t\|v_x(s)\|^2ds\\&\qquad+C\int_0^t\|u_x(s)\|_{L^4}^4ds\end{aligned}$$

with $\beta_1=\max(r+1-q,0)$.

Using lemma 3.3.2, the Sobolev embedding theorem and the Cauchy inequality, and noting the following facts

$$\int_0^t \|v_x(s)\|^2 ds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big), \tag{3.3.25}$$

and

$$\begin{aligned}\int_0^t \|u_x(s)\|_{L^4}^4 ds &\le \epsilon\int_0^t \|u_{xx}(s)\|^2 ds + C_1\int_0^t \|u_x(s)\|^2 ds\\ &\le \epsilon\int_0^t \|u_{xx}(s)\|^2 ds + C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_1},\end{aligned} \tag{3.3.26}$$

we have for small $\epsilon > 0$,

$$\begin{aligned}&\|u_x(t)\|^2 + \int_0^t\int_0^1 \left[(1+\theta^{r+1})u_x^2 + u_{xx}^2\right] dxds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_1} + C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_2}.\end{aligned}$$

The proof of this lemma is complete. □

Corollary 3.3.2. *Under assumptions of theorem 3.2.1, the following estimates hold for any* $t \in (0, +\infty)$,

$$\|v(t)\|^2 + \|u_x(t)\|^2 + \int_0^t \Big(\|u_x\|^2 + \|u_{xx}\|^2\Big)(s)ds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_2}, \tag{3.3.27}$$

$$\int_0^t\int_0^1 (1+\theta)^{r-1}\theta_x^2 dxds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_1}, \tag{3.3.28}$$

$$\int_0^t\int_0^1 (1+\theta)^{2m}u_x^2 dxds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_2} \tag{3.3.29}$$

with $\beta_1 = \max(r+1-q, 0), \beta_2 = \max(1, \beta_1)$.

Lemma 3.3.7. *Under assumptions of theorem 3.2.1, the following estimate holds for any* $t \in (0, +\infty)$,

$$\int_0^t \Big(\|u_x\|_{L^4}^4 + \|u_x\|_{L^6}^6 + \|u_x\|_{L^\infty}^4\Big)(s)ds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_2}. \tag{3.3.30}$$

Proof. The inequality (3.3.30) follows from lemma 3.3.6 and the Sobolev embedding theorem. □

Next, we establish some estimates on the velocity v.

Lemma 3.3.8. *Under assumptions of theorem 3.2.1, the following estimates hold that for any* $t \in (0, +\infty)$,

$$\|v(t)\|^2 + \int_0^t \|v_x(s)\|^2 ds \le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big), \tag{3.3.31}$$

$$\|v_x(t)\|^2 + \|u_{xx}(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+r+\beta_2}, \tag{3.3.32}$$

$$\int_0^t \|v_x(s)\|^2_{L^\infty} ds \le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+\frac{1}{2}(r+\beta_2)}. \tag{3.3.33}$$

Proof. Multiplying (3.2.7) by v in Q_t and using (3.3.19)–(3.3.21) and lemmas 3.3.5–3.3.7, we can easily derive from lemma 3.3.2 and corollary 3.3.1,

$$\|v(t)\|^2 + \int_0^t \|v_x(s)\|^2 ds \le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_2},$$

which, together with (3.3.25), implies the estimate (3.3.31).

Multiplying (3.2.7) by v_{xx}, then integrating the resulting equation over Q_t, and noting the equation (3.2.6) and the boundary condition (3.2.12), we have

$$\int_0^t\int_0^1 u_{xxx}v_{xx}\,dxds = \int_0^t\int_0^1 u_{xxx}u_{xt}\,dxds = -\int_0^t\int_0^1 u_{xxt}u_{xx}\,dxds = -\int_0^t \Big(\frac{u_{xx}^2}{2}\Big)_t ds$$

which, together with (3.3.19)–(3.3.21), lemmas 3.3.5–3.3.7 and the Cauchy inequality, implies

$$\begin{aligned}
&\|v_x(t)\|^2 + \|u_{xx}(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \\
&\le C_1 + C\int_0^t\int_0^1 \big(|u_x v_x v_{xx}| + (1+\theta^{1+r})|u_x v_{xx}| + (1+\theta^r)|\theta_x v_{xx}|\big)\,dxds \\
&\le C_1 + \frac{1}{4}\int_0^t \|v_{xx}(s)\|^2 ds + C_1\int_0^t \|v_x(s)\|^2_{L^\infty} ds \\
&\quad + C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+r}\int_0^t\int_0^1 (1+\theta)^{1+r}u_x^2\,dxds \\
&\quad + C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2(1+r)-q} \\
&\le \frac{1}{4}\int_0^t \|v_{xx}(s)\|^2 ds + C_1\Big(\int_0^t \|v_x(s)\|^2 ds\Big)^{1/2}\Big(\int_0^t \|v_{xx}(s)\|^2 ds\Big)^{1/2} \\
&\quad + C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+r+\beta_2} + C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2(1+r)-q} \\
&\le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+r+\beta_2} + \frac{1}{2}\int_0^t \|v_{xx}(s)\|^2 ds,
\end{aligned}$$

i.e.,

$$\|v_x(t)\|^2 + \|u_{xx}(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{1+r+\beta_2}.$$

Finally, by the Sobolev embedding theorem and the Hölder inequality, we can also get

$$\int_0^t \|v_x(s)\|_{L^\infty}^2 ds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{1+\frac{1}{2}(r+\beta_2)}.$$

The proof of this lemma is complete. □

Corollary 3.3.3. *Under assumptions of theorem 3.2.1, the following estimates hold for any* $t \in (0, +\infty)$,

$$\int_0^t \int_0^1 (1+\theta)^{q+r+2} v_x^2 dxds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{q+2+\frac{1}{2}(r+\beta_2)}, \tag{3.3.34}$$

$$\int_0^t \|v_x(s)\|_{L^3}^3 ds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{\beta_4}, \tag{3.3.35}$$

$$\int_0^t \int_0^1 (1+\theta)^{q+1} |v_x|^3 dxds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{\beta_5}, \tag{3.3.36}$$

$$\int_0^t \|v_x(s)\|_{L^4}^4 ds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{2+\frac{3}{2}(r+\beta_2)}, \tag{3.3.37}$$

$$\int_0^t \int_0^1 (1+\theta)^{q-r} v_x^4 dxds \le C_1 \left(1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}\right)^{\beta_6+2+\frac{3}{2}(r+\beta_2)} \tag{3.3.38}$$

where

$$\begin{cases} \beta_3 = \max\left(1+r+\beta_2, 2+\dfrac{1}{2}(r+\beta_2)\right), \\ \beta_4 = \min\left(\beta_3, \dfrac{3}{4}(2+r+\beta_2)\right), \\ \beta_5 = \min\left(\beta_4+q+1, \dfrac{3}{2}(1+r+\beta_2)\right), \\ \beta_6 = \max(q-r, 0). \end{cases}$$

Proof. Using lemma 3.3.2 and (3.3.33), we can derive

$$\begin{aligned}\int_0^t\int_0^1(1+\theta)^{q+r+2}v_x^2dxds &\le C\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q+1}\int_0^t\|v_x(s)\|_{L^\infty}^2\int_0^1(1+\theta)^{1+r}dxds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q+2+\frac{1}{2}(r+\beta_2)}.\end{aligned}$$

Using the Gagliardo–Nirenberg inequality and the Hölder inequality, we easily obtain

$$\begin{aligned}\int_0^t\int_0^1|v_x|^3dxds &\le C\int_0^t\|v_x(s)\|^{\frac{5}{2}}\|v_{xx}(s)\|^{\frac{1}{2}}ds\\ &\le C\sup_{0\le s\le t}\|v_x(s)\|\Big(\int_0^t\|v_x(s)\|^2ds\Big)^{\frac{3}{4}}\Big(\int_0^t\|v_{xx}(s)\|^2ds\Big)^{\frac{1}{4}}\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac{3}{4}(2+r+\beta_2)}.\end{aligned}$$

In addition, we also derive from (3.3.32) and the Cauchy inequality,

$$\begin{aligned}\int_0^t\int_0^1|v_x|^3dxds &\le C\sup_{0\le s\le t}\|v_x(s)\|\Big(\int_0^t\|v_x(s)\|^2ds\Big)^{\frac{3}{4}}\Big(\int_0^t\|v_{xx}(s)\|^2ds\Big)^{\frac{1}{4}}\\ &\le C\sup_{0\le s\le t}\|v_x(s)\|^2+C\Big(\int_0^t\|v_x(s)\|^2ds\Big)^{\frac{3}{2}}\Big(\int_0^t\|v_{xx}(s)\|^2ds\Big)^{\frac{1}{2}}\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_3}.\end{aligned}$$

Thus, we have the estimate (3.3.35).

Obviously, it follows from (3.3.35) that

$$\int_0^t\int_0^1(1+\theta)^{q+1}|v_x|^3dxds\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_4+q+1}.$$

On the other hand, we can also derive

$$\begin{aligned}\int_0^t\int_0^1(1+\theta)^{q+1}|v_x|^3dxds &\le C\int_0^t\int_0^1(1+\theta)^{\frac{3}{2}m}|v_x|^3dxds\\ &\le C\int_0^t\int_0^1|v_x|^3dxds+C\int_0^t\|v_x(s)\|^{\frac{5}{2}}\|v_{xx}(s)\|^{\frac{1}{2}}(V_2(s))^{\frac{3}{4}}ds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_4}\\ &\quad +C\sup_{0\le s\le t}\|v_x(s)\|^{\frac{5}{2}}\Big(\int_0^t\|v_{xx}(s)\|^2ds\Big)^{\frac{1}{4}}\Big(\int_0^t V_2(s)ds\Big)^{\frac{3}{4}}\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_4}+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac{3}{2}(1+r+\beta_2)}\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac{3}{2}(1+r+\beta_2)},\end{aligned}$$

where $V_2(t)=\int_0^1 \frac{v_x^2}{u\theta}dx$ satisfies $\int_0^{+\infty} V_2(t)dt<+\infty$. Thus we get (3.3.36).

In the same manner, we also have

$$\begin{aligned}\int_0^t\int_0^1 v_x^4 dxds &\le C\int_0^t \|v_x(s)\|^3\|v_{xx}(s)\|ds\\ &\le C\sup_{0\le s\le t}\|v_x(s)\|^2\left(\int_0^t\|v_x(s)\|^2ds\right)^{\frac12}\left(\int_0^t\|v_{xx}(s)\|^2ds\right)^{\frac12}\\ &\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{2+\frac32(r+\beta_2)}.\end{aligned}$$

Furthermore, we can get

$$\int_0^t\int_0^1(1+\theta)^{q-r}v_x^4dxds\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_6+2+\frac32(r+\beta_2)}.$$

Therefore, we complete the proof. □

Now we shall give some estimates on the absolute temperature θ.

Lemma 3.3.9. *Under assumptions of theorem 3.2.1, the following estimate holds that for any $t\in(0,+\infty)$,*

$$\left\|\theta(t)+\theta^{1+r}(t)\right\|^2+\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{10}}, \quad (3.3.39)$$

where

$$\begin{cases}\beta_7=\max(3r+3-2q,0),\\ \beta_8=\max(2r+1-2q,0),\\ \beta_9=\min\left(\beta_8,\frac12\beta_7\right),\\ \beta_{10}=\max\left(\beta_9,\beta_1+\frac12\beta_2,1+\frac12(r+\beta_2)\right).\end{cases}$$

Proof. The equation (3.2.8) can be rewritten as

$$e_t+Pv_x-\mu_0\frac{v_x^2}{u}-\left(\alpha(u,\theta)\frac{\theta_x}{u}\right)_x=0. \quad (3.3.40)$$

Multiplying (3.3.40) by e over Q_t and using lemmas 3.3.1–3.3.8, we have

$$\begin{aligned}&\left\|\theta+\theta^{r+1}\right\|^2+\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds\\ &\le C_1+C_1\int_0^t\|v_x\|_{L^\infty}^2\int_0^1(1+\theta^{r+1})dxds+C_1\int_0^t\int_0^1(1+\theta)^{2r+2}|vu_x|dxds\\ &\quad+C_1\int_0^t\int_0^1(1+\theta)^{2r+1}|v\theta_x|dxds+C_1\int_0^t\int_0^1(1+\theta)^{q+r+1}|\theta_xu_x|dxds. \quad (3.3.41)\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}
&\int_0^t\int_0^1 (1+\theta)^{2r+2}|vu_x|dxds\\
&\quad\leq\left(\int_0^t\int_0^1 (1+\theta)^{2r+2}u_x^2dxds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1 (1+\theta)^{2r+2}v^2dxds\right)^{\frac{1}{2}}\\
&\quad\leq C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\beta_1}\left(\int_0^t\int_0^1 (1+\theta)^{2r+2}u_x^2dxds\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_0^t\int_0^1 (1+\theta)^{2m}v^2dxds\right)^{\frac{1}{2}}\\
&\quad\leq C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\beta_1+\frac{1}{2}\beta_2}. \qquad (3.3.42)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\int_0^t\int_0^1 (1+\theta)^{2r+1}|v\theta_x|dxds&\leq\left(\int_0^t V_1(s)\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1\frac{\theta^2(1+\theta)^{4r+2}v^2}{1+\theta^q}dxds\right)^{\frac{1}{2}}\\
&\leq C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\frac{1}{2}\beta_7}\left(\int_0^t\int_0^1 (1+\theta)^{2m}v^2dxds\right)^{\frac{1}{2}}\\
&\leq C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\frac{1}{2}\beta_7}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_0^1 (1+\theta)^{2r+1}|v\theta_x|dxds&\leq\frac{1}{4}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2dxds\\
&\quad+C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\beta_8}\int_0^t\int_0^1 (1+\theta)^{2m}v^2dxds\\
&\leq\frac{1}{4}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2dxds+C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\beta_8}.
\end{aligned}$$

So

$$\int_0^t\int_0^1 (1+\theta)^{2r+1}|v\theta_x|dxds\leq\frac{1}{4}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2dxds+C_1\left(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{\beta_9}. \qquad (3.3.43)$$

On the other hand, by the Cauchy inequality and corollary 3.3.2, we have

$$\int_0^t\int_0^1 (1+\theta)^{q+r+1}|\theta_x u_x|dxds \le \frac{1}{4}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2 dxds + C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_2+1}. \tag{3.3.44}$$

Therefore, inserting (3.3.42)–(3.3.44) into (3.3.41), we can derive from lemmas 3.3.1–3.3.8 that

$$\begin{aligned}
&\left\|\theta+\theta^{1+r}\right\|^2 + \int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2 dxds\\
&\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{1+\frac{1}{2}(r+\beta_2)} + C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_1+\frac{1}{2}\beta_2}\\
&\quad + C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_9} + C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_2+1}\\
&\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{10}}
\end{aligned}$$

which implies the estimate (3.3.39). □

Lemma 3.3.10. *Under assumptions of theorem 3.2.1, the following estimate holds that for any* $t\in(0,+\infty)$,

$$\int_0^1 (1+\theta)^{2q}\theta_x^2 dx + \int_0^t\int_0^1 (1+\theta)^{q+r}\theta_t^2 dxds \le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{17}}, \tag{3.3.45}$$

where

$$\begin{cases}
\beta_{11}=\max(3q+2-r,0),\ \beta_{12}=\max(\beta_{10}+\beta_{11},1+r+\beta_2),\\
\beta_{13}=\min\left(\beta_{12},1+\dfrac{1}{2}(r+\beta_2+\beta_{10}+\beta_{11})\right),\ \beta_{14}=\max\left(2\beta_6+2\beta_2+\beta_{10},2+\beta_6+\dfrac{3}{2}(r+\beta_2)\right),\\
\beta_{15}=\max\left(2\beta_6+2\beta_2+r+2,\beta_2+\beta_6+q+2,2+\beta_6+\dfrac{3}{2}(r+\beta_2)\right),\ \beta_{16}=\min(\beta_{14},\beta_{15}),\\
\beta_{17}=\max\left(2+\beta_6+\dfrac{3}{2}(r+\beta_2),\beta_{13},\beta_{16}\right).
\end{cases}$$

Proof. Let

$$H(x,t)=H(u,\theta)=\int_0^\theta \frac{\alpha(u,\xi)}{u}d\xi,\ X(t)=\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_t^2 dxds.$$

Then we easily obtain that

$$H_t=H_u v_x+\frac{\alpha\theta_t}{u},\ H_{xt}=\left(\frac{\alpha\theta_x}{u}\right)_t+H_u v_{xx}+H_{uu}v_x u_x+\left(\frac{\alpha}{u}\right)_u u_x\theta_t.$$

Multiplying the equation (3.2.8) by H_t over Q_t, we can get

$$\int_0^t\int_0^1\left(e_\theta\theta_t+\theta P_\theta v_x-\frac{\mu_0 v_x^2}{u}\right)H_t dxds+\int_0^t\int_0^1\frac{\alpha\theta_x H_{tx}}{u}dxds=0. \tag{3.3.46}$$

Obviously, we can derive from (3.2.25) and (3.2.26) that

$$|H_u|+|H_{uu}|\le C(1+\theta)^{q+1}. \tag{3.3.47}$$

Now we need to estimate each term in (3.3.46). We easily derive from (3.2.21), (3.2.25) and lemma 3.3.5 that

$$\int_0^t\int_0^1 e_\theta\theta_t H_t dxds\ge CX(t)-C\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{2+q+\frac{1}{2}(r+\beta_2)}. \tag{3.3.48}$$

Similarly,

$$\begin{aligned}
&\left|\int_0^t\int_0^1\left(\theta P_\theta v_x-\frac{\mu_0 v_x^2}{u}\right)H_t dxds\right|\\
&\quad\le\frac{C}{8}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{2+q+\frac{1}{2}(r+\beta_2)}+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_5}\\
&\qquad+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{2+\beta_6+\frac{3}{2}(r+\beta_2)}\\
&\quad\le\frac{C}{8}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{2+\beta_6+\frac{3}{2}(r+\beta_2)},
\end{aligned} \tag{3.3.49}$$

$$\int_0^t\int_0^1\frac{\alpha\theta_x}{u}\left(\frac{\alpha\theta_x}{u}\right)_t dxds\ge C_1\int_0^1(1+\theta)^{2q}\theta_x^2dx-C_1 \tag{3.3.50}$$

and

$$\begin{aligned}
&\left|\int_0^t\int_0^1\frac{\alpha\theta_x}{u}(H_u v_{xx}+H_{uu}v_x u_x)dxds\right|\\
&\quad\le C\int_0^t\int_0^1(1+\theta)^{2q+1}(|\theta_x v_{xx}|+|u_x v_x\theta_x|)dxds\\
&\quad\le C\int_0^t\int_0^1(1+\theta)^{4q+2}\theta_x^2dxds+C\int_0^t\int_0^1 v_{xx}^2dxds+C\int_0^t\int_0^1 u_x^2v_x^2dxds\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{11}}\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{1+r+\beta_2}\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{12}}.
\end{aligned}$$

On the other hand, using the Hölder inequality, we can derive

$$\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}(H_u v_{xx} + H_{uu}v_x u_x)\,dxds\right| \\
&\quad\le C_1\int_0^t\int_0^1 (1+\theta)^{2q+1}(|\theta_x v_{xx}| + |u_x v_x\theta_x|)\,dxds \\
&\quad\le C_1\left(\int_0^t\int_0^1 (1+\theta)^{4q+2}\theta_x^2\,dxds\right)^{\frac{1}{2}}\left[\left(\int_0^t \|v_{xx}(s)\|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^t \|u_x(s)v_x(s)\|^2 ds\right)^{\frac{1}{2}}\right] \\
&\quad\le C_1\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{1+\frac{1}{2}(r+\beta_2+\beta_{10}+\beta_{11})}.
\end{aligned}$$

Therefore, we can derive from the above two estimates that

$$\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}(H_u v_{xx} + H_{uu}v_x u_x)\,dxds\right| \le C_1\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_{13}}. \tag{3.3.51}$$

Applying the Young inequality and lemmas 3.3.1–3.3.9, we can get

$$\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}\left(\frac{\alpha}{u}\right)_u u_x\theta_t\,dxds\right| \\
&\quad\le C\int_0^t\int_0^1 (1+\theta)^q\left|\frac{\alpha\theta_x}{u}u_x\theta_t\right| dxds \\
&\quad\le \frac{C}{8}X(t) + C_1\int_0^t\int_0^1 \left(\frac{\alpha\theta_x}{u}\right)^2(1+\theta)^{q-r}u_x^2\,dxds \\
&\quad\le \frac{C}{8}X(t) + C_1\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_6+\beta_2}\int_0^t \left\|\frac{\alpha\theta_x}{u}\right\|_{L^\infty}^2 ds \\
&\quad\le \frac{C}{8}X(t) + C_1\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_6+\beta_2}\int_0^t\left(\left\|\frac{\alpha\theta_x}{u}\right\|^2 + \int_0^1\left|\frac{\alpha\theta_x}{u}\left(\frac{\alpha\theta_x}{u}\right)_x\right| dx\right) ds \\
&\quad\le \frac{C}{8}X(t) + C_1\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_6+\beta_2}\left\{\left(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta_6}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2\,dxds\right. \\
&\qquad\left. + \left(\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2\,dxds\right)^{1/2}\left(\int_0^t\int_0^1 (1+\theta)^{q-r}\left|\left(\frac{\alpha\theta_x}{u}\right)_x\right|^2 dxds\right)^{1/2}\right\}
\end{aligned}$$

which, together with (3.3.6), implies

$$\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}\Big(\frac{\alpha}{u}\Big)_u u_x\theta_t\,dxds\right|\\
&\quad\le \frac{C}{8}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2\beta_6+\beta_2+\beta_{10}}+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+\beta_2+\frac12\beta_{10}}\\
&\qquad\times\left\{X(t)+\int_0^t\int_0^1\Big((1+\theta)^{q+r+2}v_x^2+(1+\theta)^{q-r}v_x^4\Big)dxds\right\}^{1/2}\\
&\quad\le \frac{C}{4}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2\beta_6+2\beta_2+\beta_{10}}+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2+q+\frac12(r+\beta_2)}\\
&\qquad+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2+\beta_6+\frac12(r+\beta_2)}\\
&\quad\le \frac{C}{4}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_{14}}. \qquad (3.3.52)
\end{aligned}$$

However, we can also derive that

$$\begin{aligned}
&\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}\Big(\frac{\alpha}{u}\Big)_u u_x\theta_t\,dxds\right|\\
&\quad\le \frac{C}{8}X(t)+C\int_0^t\int_0^1\Big(\frac{\alpha\theta_x}{u}\Big)^2(1+\theta)^{q-r}u_x^2\,dxds\\
&\quad\le \frac{C}{8}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+\beta_2}\\
&\qquad\times\left\{\int_0^t\int_0^1(1+\theta)^{2q}\theta_x^2\,dxds+\int_0^t\int_0^1(1+\theta)^q|\theta_x|\left|\Big(\frac{\alpha\theta_x}{u}\Big)_x\right|dxds\right\}\\
&\quad\le \frac{C}{8}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+\beta_2}\left\{\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q+2}\int_0^t V_1(s)\,ds.\right.\\
&\qquad\left.+\Big(\int_0^t V_1(s)\,ds\Big)^{1/2}\left(\int_0^t\int_0^1\theta^2(1+\theta)^q\left|\Big(\frac{\alpha\theta_x}{u}\Big)_x\right|^2dxds\right)^{1/2}\right\}\\
&\quad\le \frac{C}{8}X_1(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+q+2+\beta_2}\\
&\qquad+C\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+\beta_2+\frac{r+2}{2}}\left(\int_0^t\int_0^1(1+\theta)^{q-r}\left|\Big(\frac{\alpha\theta_x}{u}\Big)_x\right|^2dxds\right)^{1/2}\\
&\quad\le \frac{C}{4}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_6+\beta_2+q+2}+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2\beta_6+2\beta_2+r+2}\\
&\qquad+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2+q+\frac12(r+\beta_2)}+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2+\beta_6+\frac12(r+\beta_2)}\\
&\quad\le \frac{C}{4}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_{15}}
\end{aligned}$$

which, along with (3.3.52), leads to

$$\left|\int_0^t\int_0^1 \frac{\alpha\theta_x}{u}\Big(\frac{\alpha}{u}\Big)_u u_x\theta_t dxds\right| \leq \frac{C}{4}X(t)+C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_{16}}. \tag{3.3.53}$$

Therefore, inserting (3.3.48)–(3.3.53) into (3.3.46), we can obtain (3.3.45). □

The next lemma shows us the global existence of solutions in space H^1_+.

Lemma 3.3.11. *Under assumptions of theorem 3.2.1, the following estimates hold that for any $t\in(0,+\infty)$,*

$$\|\theta(t)\|_{L^\infty}\leq C_1, \tag{3.3.54}$$

$$\|\theta_x(t)\|^2+\|u_x(t)\|^2+\|v_x(t)\|^2+\|u_{xx}(t)\|^2 + \int_0^t\Big(\|u_x\|^2+\|\theta_x\|^2+\|\theta_t\|^2+\|v_x\|^2+\|u_{xx}\|^2+\|v_{xx}\|^2\Big)(s)\leq C_1. \tag{3.3.55}$$

Proof. Similarly to the proof of lemma 2.3.9, using the Sobolev embedding theorem, lemmas 3.3.2–3.3.10, and the Young inequality, we can obtain

$$\|\theta(t)\|^{2q+r+3}_{L^\infty}\leq C_1+C_1\int_0^1(1+\theta)^{2q}\theta_x^2dx\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_{17}}. \tag{3.3.56}$$

Similarly,

$$\|\theta(t)\|\leq C_1+C_1\int_0^1|\theta^{q+1}\theta_x|dx\leq C_1+C_1\Big(\int_0^1(1+\theta)^{2q}\theta_x^2dx\Big)^{1/2}\Big(\int_0^1\theta^{2r+2}dx\Big)^{1/2}$$

which implies

$$\|\theta(t)\|^{2q+2r+4}_{L^\infty}\leq C_1+C_1\int_0^1(1+\theta)^{2q}\theta_x^2dx\int_0^1\theta^{2r+2}dx\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{\beta_{10}+\beta_{17}}. \tag{3.3.57}$$

After a lengthy calculation, we can derive from assumptions (3.2.18)–(3.2.20) that $\beta_{10}+\beta_{17}<2q+2r+4$ or $\beta_{17}<2q+r+3$. Therefore, using the Young inequality, we derive from (3.3.56) or (3.3.57) that

$$\|\theta(t)\|_{L^\infty}\leq C_1,$$

which, along with lemmas 3.3.6–3.3.10, implies the desired estimate (3.3.55). □

3.3.2 Asymptotic Behavior in H^1_+

The next two lemmas are aimed at showing the asymptotic behavior of global solutions to the problem (3.2.6)–(3.2.12) in H^1_+.

Lemma 3.3.12. *Under assumptions of theorem 3.2.1, the following estimates hold for any* $t \in (0, +\infty)$,

$$\frac{d}{dt}\|u_x(t)\|^2 \leq \|v_{xx}(t)\|^2 + \|u_x(t)\|^2, \tag{3.3.58}$$

$$\frac{d}{dt}\|\theta_x(t)\|^2 + C\int_0^1 (1+\theta)^{q-r}\theta_{xx}^2\,dx \leq C_1\Big(\|v_{xx}(t)\|^2 + 1\Big), \tag{3.3.59}$$

$$\|\theta_x(t)\|^2 + \int_0^t\int_0^1 (1+\theta)^{q-r}\theta_{xx}^2\,dxds \leq C_1, \tag{3.3.60}$$

$$\frac{d}{dt}\Big(\|v_x(t)\|^2 + \|u_{xx}(t)\|^2\Big) \leq C_1\Big(\|v_{xx}(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2\Big). \tag{3.3.61}$$

Proof. Inequalities (3.3.58)–(3.3.60) follow from lemma 2.3.11 (see also lemma 2.1.13 in Qin [99]). Multiplying (3.2.7) by u_{xt}, noting that $u_{xt} = v_{xx}$, using the embedding theorem and (3.3.55), we can easily obtain (3.3.61). □

Lemma 3.3.13. *Under assumptions of theorem 3.2.1, we have as* $t \to +\infty$,

$$\|u(t) - \overline{u_0}\|_{H^2} \to 0, \|u(t) - \overline{u_0}\|_{H^1} \to 0, \|u_x(t)\| \to 0, \|u(t) - \overline{u_0}\|_{L^\infty} \to 0, \tag{3.3.62}$$

$$\|v(t)\|_{H^1} \to 0, \|v_x(t)\| \to 0, \tag{3.3.63}$$

$$\|\theta_x(t)\| \to 0, \tag{3.3.64}$$

$$\|\theta(t) - \bar{\theta}\|_{H^1} \to 0, \|\theta(t) - \bar{\theta}\|_{L^\infty} \to 0. \tag{3.3.65}$$

Proof. Applying the Shen–Zheng inequality (theorem 1.2.10), we derive from lemmas 3.3.11 and 3.3.12 as $t \to +\infty$,

$$\|u_x(t)\| \to 0,\ \|\theta_x(t)\| \to 0,\ \|v_x(t)\| \to 0,\ \|u_{xx}(t)\| \to 0. \tag{3.3.66}$$

Furthermore, (3.3.62)–(3.3.64) follow from the embedding theorem and the estimate (3.3.4).

Noting the estimate (3.3.2), lemmas 3.3.1–3.3.12 and the Poincaré inequality, we can conclude

$$\begin{aligned}\|e(u,\theta) - e(\bar{u},\bar{\theta})\| &\leq \left\|e(u,\theta) - \int_0^1 e(u,\theta)\,dx\right\| + \frac{\|v(t)\|^2}{2} + \kappa\frac{\|u_x(t)\|^2}{2}\\ &\leq C_1(\|u_x(t)\| + \|\theta_x(t)\| + \|v_x(t)\|)\end{aligned} \tag{3.3.67}$$

with $\bar{u} = \int_0^1 u dx = \int_0^1 u_0 dx = \overline{u_0}$. By the mean value theorem, we know that there exist $\tilde{u}$ and $\tilde{\theta}$ with $\tilde{u} = \lambda\theta + (1-\lambda)\bar{\theta}$ and $\tilde{u} = \lambda u + (1-\lambda)\bar{u}$ such that

$$e(u,\theta) - e(\bar{u},\bar{\theta}) = e_u(\tilde{u},\tilde{\theta})(u-\bar{u}) + e_\theta(\tilde{u},\tilde{\theta})(\theta-\bar{\theta}). \tag{3.3.68}$$

Thus we derive from lemmas 3.3.1–3.3.12 that

$$0 < C_1^{-1} \le \min(u,\bar{u}) \le \tilde{u} \le C_1, \quad 0 < \min(\theta,\bar{\theta}) \le \tilde{\theta} \le C_1$$

which, along with (3.3.66) and (3.3.68), gives us that as $t \to +\infty$,

$$\begin{aligned}\|\theta-\bar{\theta}\| &\le \left\|e_\theta^{-1}(\tilde{u},\tilde{\theta})\left[e(u,\theta)-e(\bar{u},\bar{\theta})\right]\right\| + \left\|e_\theta^{-1}(\tilde{u},\tilde{\theta})e_u(\tilde{u},\tilde{\theta})(u-\bar{u})\right\| \\ &\le C_1(\|u_x\|+\|\theta_x\|+\|v_x\|) \to 0.\end{aligned} \tag{3.3.69}$$

Therefore, (3.3.65) follows from (3.3.66) and (3.3.69). ☐

Lemma 3.3.14. *Under assumptions of theorem 3.2.1, the unique generalized global solution* $(u(t), v(t), \theta(t))$ *to the problem (3.2.6)–(3.2.12) in* H_+^1 *defines a nonlinear* C_0*-semigroup* $S(t)$ *on* H_+^1 *and satisfies for any* $(u_0, v_0, \theta_0) \in H_+^1$,

$$(u(t), v(t), \theta(t)) = S(t)(u_0, v_0, \theta_0) \in C([0, +\infty), H_+^1). \tag{3.3.70}$$

Proof. By lemmas 3.3.11 and 3.3.12, we know that for any $t > 0$, the operator $S(t) : (u_0, v_0, \theta_0) \in H_+^1 \to (u(t), v(t), \theta(t)) \in H_+^1$ exists and satisfies on H_+^1 for any $t_1,\ t_2 \in [0, +\infty)$,

$$S(t_1+t_2) = S(t_1)S(t_2) = S(t_2)S(t_1). \tag{3.3.71}$$

Moreover, $S(t)$ is uniformly bounded on H_+^1 with respect to $t > 0$, *i.e.*,

$$\|S(t)\|_{\mathcal{L}(H_+^1, H_+^1)} \le C_1. \tag{3.3.72}$$

Now we verify the continuity of $S(t)$ with respect to the initial data in H_+^1 for any fixed $t > 0$. Assume that $(u_{0j}, v_{0j}, \theta_{0j}) \in H_+^1$, $(u_j,\ v_j,\ \theta_j) = S(t)(u_{0j},\ v_{0j},\ \theta_{0j})(j = 1, 2)$, and $(u,\ v,\ \theta) = (u_1,\ v_1,\ \theta_1) - (u_2,\ v_2,\ \theta_2)$. Subtracting the corresponding equations (3.2.6)–(3.2.8) satisfied by $(u_1,\ v_1,\ \theta_1)$ and $(u_2,\ v_2,\ \theta_2)$, we obtain

$$\begin{cases} u_t = v_x, & (3.3.73)\\ v_t = -P_u(u_1,\theta_1)u_x - (P_u(u_1,\theta_1) - P_u(u_2,\theta_2))u_{2x} - P_\theta(u_1,\theta_1)\theta_x & \\ \quad - (P_\theta(u_1,\theta_1) - P_\theta(u_2,\theta_2))\theta_{2x} + \mu_0\left[\dfrac{v_{xx}}{u_1} - \dfrac{v_x u_{1x}}{u_1^2} - \left(\dfrac{v_{2x}u}{u_1u_2}\right)_x\right] - \kappa u_{xxx}, & (3.3.74)\\ e_\theta(u_1,\theta_1)\theta_t = -(e_\theta(u_1,\theta_1) - (e_\theta(u_2,\theta_2))\theta_{2t} - (e_u(u_1,\theta_1) - e_u(u_2,\theta_2))v_{2x} & \\ \quad - e_u(u_1,\theta_1)v_x - P(u_1,\theta_1)v_x - (P(u_1,\theta_1) - P(u_2,\theta_2))v_{2x} & \\ \quad + \left[\dfrac{\alpha(u_1,\theta_1)\theta_x}{u_1} + \left(\dfrac{\alpha(u_1,\theta_1)}{u_1} - \dfrac{\alpha(u_2,\theta_2)}{u_2}\right)\theta_{2x}\right]_x, & (3.3.75)\end{cases}$$

with

$$\begin{aligned}&t=0: u=u_0, \quad v=v_0, \quad \theta=\theta_0,\\&x=0,1: u_x=0, \quad v=0, \quad \theta_x=0.\end{aligned} \tag{3.3.76}$$

By lemmas 3.3.11 and 3.3.12, we easily get

$$\begin{aligned}&\frac{d}{dt}\left(\|u(t)\|^2+\|v(t)\|^2+\left\|\sqrt{e_\theta(u_1,\theta_1)}\theta(t)\right\|^2\right)+C_1^{-1}\left(\|v_x(t)\|^2+\|\theta_x(t)\|^2\right)\\&\quad\le C_1H_1(t)\left(\|u(t)\|^2+\|\theta(t)\|_{H^1}^2\right)\end{aligned} \tag{3.3.77}$$

where $H_1(t)=\|\theta_{1t}(t)\|^2+\|\theta_{2t}(t)\|^2+\|v_{1xx}(t)\|^2+\|v_{2xx}(t)\|^2+\|\theta_{1xx}(t)\|^2+\|\theta_{2xx}(t)\|^2$ $+1$ satisfies for any $t>0$,

$$\int_0^t H_1(s)ds\le C_1(1+t). \tag{3.3.78}$$

Differentiating (3.3.73) with respect to x, multiplying the resultant by u_x and integrating by parts, and using the Cauchy inequality, we derive that for any small $\delta>0$,

$$\frac{d}{dt}\|u_x(t)\|^2\le\delta\|v_{xx}(t)\|^2+C(\delta)\|u_x(t)\|^2. \tag{3.3.79}$$

Multiplying (3.3.74) by v_{xx}, integrating it over [0, 1] with respect to x, and using lemma 3.3.11, the embedding theorem and the mean value theorem, we obtain for any small $\varepsilon>0$,

$$\begin{aligned}&\frac{d}{dt}\left(\|u_{xx}(t)\|^2+\|v_x(t)\|^2\right)+C_1^{-1}\|v_{xx}(t)\|^2\\&\quad\le\varepsilon\|v_{xx}(t)\|^2+C_1(\varepsilon)H_1(t)\left(\|v_x(t)\|^2+\|u(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2\right).\end{aligned} \tag{3.3.80}$$

Similarly, by the equation (3.3.75), we also have

$$\begin{aligned}&\frac{d}{dt}\left\|\sqrt{\alpha(u_1,\theta_1)}\theta_x(t)\right\|^2+C_1^{-1}\|\theta_t(t)\|^2\\&\quad\le C_1H_1(t)\left(\|v_x(t)\|^2+\|u(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2\right).\end{aligned} \tag{3.3.81}$$

Adding up (3.3.77) and (3.3.79)–(3.3.81), and taking $\delta>0$ and $\varepsilon>0$ small enough, we finally get

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_1(t)&\le C_1H_1(t)\left(\|v_x(t)\|^2+\|u(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2\right)\\&\le C_1H_1(t)\mathcal{M}_1(t)\end{aligned} \tag{3.3.82}$$

where

$$\begin{aligned}\mathcal{M}_1(t) &= \|u(t)\|^2 + \|u_x(t)\|^2 + \|v(t)\|^2 + \|v_x(t)\|^2 + \|u_{xx}(t)\|^2\\ &\quad + \left\|\sqrt{e_\theta(u_1,\theta_1)}\theta(t)\right\|^2 + \left\|\sqrt{\alpha(u_1,\theta_1)}\theta_x(t)\right\|^2\end{aligned}$$

satisfies

$$\begin{aligned}C_1^{-1}\Big(\|v(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^1}^2\Big) &\le \mathcal{M}_1(t)\\ &\le C_1\Big(\|v(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^1}^2\Big).\end{aligned} \tag{3.3.83}$$

Thus (3.3.82), combined with the Gronwall inequality, (3.3.78) and (3.3.83), implies for any fixed $t > 0$,

$$\begin{aligned}\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 &\le C_1\mathcal{M}_1(0)\exp\left(C_1\int_0^t H_1(s)\,ds\right)\\ &\le C_1\exp(C_1 t)\Big(\|u_0\|_{H^2}^2 + \|v_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2\Big),\end{aligned}$$

that is,

$$\begin{aligned}&\|S(t)(u_{01},v_{01},\theta_{01}) - S(t)(u_{02},v_{02},\theta_{02})\|_{H_+^1}\\ &\quad\le C_1\exp(C_1 t)\|(u_{01},v_{01},\theta_{01}) - (u_{02},v_{02},\theta_{02})\|_{H_+^1}\end{aligned} \tag{3.3.84}$$

which leads to the continuity of $S(t)$ with respect to the initial data in H_+^1. By (3.3.71) and (3.3.72), we know that only if we prove that for any $(u_0, v_0, \theta_0) \in H_+^1$, as $t \to 0^+$,

$$\|S(t)(u_0,v_0,\theta_0) - (u_0,v_0,\theta_0)\|_{H_+^1} \to 0 \tag{3.3.85}$$

which also leads to

$$S(0) = I, \tag{3.3.86}$$

with I being the unit operator in H_+^1, we can also obtain (3.3.70). Indeed, we can choose a function sequence $(u_0^m, v_0^m, \theta_0^m)$ which is smooth enough such that as $m \to +\infty$,

$$\left\|(u_0^m, v_0^m, \theta_0^m) - (u_0, v_0, \theta_0)\right\|_{H_+^1} \to 0. \tag{3.3.87}$$

By the regularity results, we know that for arbitrary but fixed $T > 0$, and $(x, t) \in (0, 1) \times (0, T)$, there exists a unique global smooth solution (u^m, v^m, θ^m) in H_+^1. Thus we have, for $m = 1, 2, 3,\ldots$ and as $t \to 0^+$,

$$\left\|(u^m(t), v^m(t), \theta^m(t)) - (u_0^m, v_0^m, \theta_0^m)\right\|_{H_+^1} \to 0. \tag{3.3.88}$$

Fixing $T = 1$, by the continuity of the operator $S(t)$, (3.3.84) and (3.3.87), for any $t \in [0, 1]$ and as $m \to +\infty$,

$$\begin{aligned}\|(u^m(t), v^m(t), \theta^m(t)) - (u(t), v(t), \theta(t))\|_{H^1_+} &= \left\|S(t)(u_0^m, v_0^m, \theta_0^m) - S(t)(u_0, v_0, \theta_0)\right\|_{H^1_+} \\ &\le C_1\left\|(u_0^m, v_0^m, \theta_0^m) - (u_0, v_0, \theta_0)\right\|_{H^1_+} \to 0,\end{aligned}$$

which, together with (3.3.87) and (3.3.88), yields, as $m \to +\infty$ and $t \to 0^+$,

$$\begin{aligned}\|S(t)(u_0, v_0, \theta_0) - (u_0, v_0, \theta_0)\|_{H^1_+} &= \|(u(t), v(t), \theta(t)) - (u_0, v_0, \theta_0)\|_{H^1_+} \\ &\le \|(u^m(t), v^m(t), \theta^m(t)) - (u(t), v(t), \theta(t))\|_{H^1_+} \\ &\quad + \left\|(u^m(t), v^m(t), \theta^m(t)) - (u_0^m, v_0^m, \theta_0^m)\right\|_{H^1_+} \\ &\quad + \left\|(u_0^m, v_0^m, \theta_0^m) - (u_0, v_0, \theta_0)\right\|_{H^1_+} \to 0.\end{aligned}$$

Thus we obtain (3.3.85) and (3.3.86). The proof is complete. □

The following lemma gives the uniform global (in time) positive lower boundedness (independent of t) of the absolute temperature θ.

Lemma 3.3.15. *If assumptions of theorem 3.2.1 hold and the initial data* $(u_0, v_0, \theta_0) \in H^1_+$, *then the generalized global solution* $(u(t), v(t), \theta(t))$ *to the problem (3.2.6)–(3.2.12) satisfies for any* $(x, t) \in [0, 1] \times [0, +\infty)$,

$$0 < C_1^{-1} \le \theta(x, t). \tag{3.3.89}$$

Proof. By contradiction, if (3.3.89) is not true, that is,

$$\inf_{(x,t)\in[0,1]\times[0,+\infty)} \theta(x, t) = 0,$$

then there exists a sequence $(x_n, t_n) \in [0, 1] \times [0, +\infty)$ such that

$$\lim_{n\to+\infty} \theta(x_n, t_n) = 0. \tag{3.3.90}$$

However, if the sequence t_n has a subsequence, also denoted by t_n, converging to $+\infty$, then by the asymptotic behavior results in lemma 3.3.13, we have that

$$\lim_{n\to+\infty} \theta(x_n, t_n) = \bar{\theta} > 0$$

which contradicts (3.3.90). If the sequence t_n is bounded, then there exists a constant $N > 0$, independent of n, such that for any $n = 1, 2, 3, \ldots, 0 < t_n < N$. Thus there is a point $(x^*, t^*) \in [0, 1] \times [0, N]$ such that $(x_n, t_n) \to (x^*, t^*)$ as $n \to +\infty$. On the other hand, by (3.3.90) and the continuity of solutions, we know that as $n \to +\infty$,

$$\lim_{n\to+\infty} \theta(x_n, t_n) = \theta(x^*, t^*) = 0,$$

which contradicts (3.3.1). Thus the proof is complete. □

3.3.3 Exponential Stability in H_+^1

In this subsection, we shall show the exponential stability of C_0-semigroup $S(t)$, *i.e.*, (3.2.28). Similarly to the assertion in Qin [99], let the density of the fluids $\rho = \frac{1}{u}$, then the entropy $\eta = \eta\left(\frac{1}{\rho}, \theta\right)$ naturally satisfies

$$\frac{\partial\eta}{\partial\rho} = -\frac{P_\theta}{\rho^2}, \ \frac{\partial\eta}{\partial\theta} = \frac{e_\theta}{\theta}. \tag{3.3.91}$$

Define the transform

$$\mathcal{A} : (\rho, \theta) \in \mathcal{D}_{\rho,\theta} = \{(\rho, \theta) : \rho > 0, \theta > 0\} \to (u, \eta) \in \mathcal{A}\mathcal{D}_{\rho,\theta}, \tag{3.3.92}$$

we know that there exists a unique function $\theta = \theta(u, \eta)$ as the smooth function of $(u, \eta) \in \mathcal{A}\mathcal{D}_{\rho,\theta}$. Thus functions e, P can be also regarded as the smooth function of (u, η), then e and P satisfy

$$e_u = -P, \ e_\eta = \theta, \ P_u = -\rho^2 P_\rho - \frac{\theta P_\theta^2}{e_\theta}, \ P_\eta = \frac{\theta P_\theta}{e_\theta}, \ \theta_u = -\frac{\theta P_\theta}{e_\theta}, \ \theta_\eta = \frac{\theta}{e_\theta}. \tag{3.3.93}$$

We define the following energy form

$$\mathcal{E}(u, v, \eta, u_x) = \frac{1}{2}v^2 + \frac{\kappa}{2}u_x^2 + e(u, \eta) - e(\overline{u}, \overline{\eta}) - \frac{\partial e}{\partial u}(\overline{u}, \overline{\eta})(u - \overline{u}) - \frac{\partial e}{\partial\eta}(\overline{u}, \overline{\eta})(\eta - \overline{\eta}), \tag{3.3.94}$$

where $\overline{\varrho} = \frac{1}{\overline{u}}, \overline{\eta} = \eta\left(\frac{1}{\overline{\rho}}, \overline{\theta}\right)$ and $\overline{u}, \overline{\theta}$ are the same as in (3.2.29) and (3.2.30).

The next lemmas show the exponential stability of the generalized global solution $(u(t), v(t), \theta(t))$ in H_+^1 (or, equivalently, of the C_0-semigroup $S(t)$ on H_+^1).

Lemma 3.3.16. *Under assumptions of theorem 3.2.1, for the unique generalized global solution $(u(t), v(t), \theta(t))$ to the problem (3.2.6)–(3.2.12), the following estimate holds*

$$\begin{aligned}\frac{1}{2}v^2 + \frac{\kappa}{2}u_x^2 + C_1^{-1}(|u - \overline{u}|^2 + |\eta - \overline{\eta}|^2) &\le \mathcal{E}(u, v, \eta, u_x) \\ &\le \frac{1}{2}v^2 + \frac{\kappa}{2}u_x^2 + C_1(|u - \overline{u}|^2 + |\eta - \overline{\eta}|^2). \end{aligned}\tag{3.3.95}$$

Proof. Similarly to the proof of lemma 2.3.13, we can obtain (3.3.95). □

Lemma 3.3.17. *Under assumptions of theorem 3.2.1, there exists a positive constant $\gamma_1' = \gamma_1'(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_1']$, the generalized global solution $(u(t), v(t), \theta(t))$ to the problem (3.2.6)–(3.2.12) satisfies for any $t \in (0, +\infty)$,*

$$\begin{aligned}&e^{\gamma t}\Big(\|v(t)\|^2 + \|u(t) - \overline{u}\|^2 + \left\|\theta(t) - \overline{\theta}\right\|^2 + \|u_x(t)\|^2 + \|\rho_x(t)\|^2 + \|u_{xx}(t)\|^2\Big) \\ &\quad + \int_0^t e^{\gamma s}\Big(\|u_x\|^2 + \|\rho_x\|^2 + \|\theta_x\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 + \|u_{xx}\|^2\Big)(s)\,ds \le C_1. \end{aligned}\tag{3.3.96}$$

Proof. We easily derive from equations (3.2.6)–(3.2.8) that $(\rho,\ v,\ \eta)$ satisfies

$$\left(e+\frac{1}{2}v^2+\frac{\kappa}{2}u_x^2\right)_t=(-Pv+\mu_0\rho vv_x+\alpha\rho\theta_x-\kappa u_{xx}v+\kappa u_xv_x)_x, \tag{3.3.97}$$

$$\eta_t=\left(\frac{\alpha\rho\theta_x}{\theta}\right)_x+\mu_0\rho\frac{v_x^2}{\theta}+\alpha\rho\left(\frac{\theta_x}{\theta}\right)^2. \tag{3.3.98}$$

Noting that $\overline{u}_t=0$, $\overline{\theta}_t=0$, we derive from the equations (3.3.97), (3.3.98), (3.2.6) and (3.2.7) that

$$\begin{aligned}&\mathcal{E}_t\left(\frac{1}{\rho},v,\eta,u_x\right)+\frac{\overline{\theta}}{\theta}\left(\mu_0\rho v_x^2+\alpha\rho\frac{\theta_x^2}{\theta}\right)\\&\quad=\left[\mu_0\rho vv_x+\alpha\left(1-\frac{\overline{\theta}}{\theta}\right)\rho\theta_x-\left(P-P\left(\frac{1}{\overline{\rho}},\overline{\theta}\right)\right)v+\kappa u_xv_x-\kappa u_{xx}v\right]_x\end{aligned} \tag{3.3.99}$$

and

$$\left[\frac{1}{2}\mu_0^2\left(\frac{\rho_x}{\rho}\right)^2+\mu_0v\frac{\rho_x}{\rho}\right]_t+\mu_0P_\rho\frac{\rho_x^2}{\rho}+\kappa\mu_0u_{xxx}\frac{\rho_x}{\rho}=-\mu_0P_\theta\frac{\rho_x\theta_x}{\rho}-\mu_0(\rho vv_x)_x+\mu_0\rho v_x^2. \tag{3.3.100}$$

Multiplying (3.3.99), (3.3.100) by $e^{\gamma t}$, $\beta e^{\gamma t}$, respectively, and then adding the results up, we can get

$$\begin{aligned}&\frac{d}{dt}G(t)+e^{\gamma t}\left[\frac{\overline{\theta}}{\theta}\left(\mu_0\rho v_x^2+\alpha\rho\frac{\theta_x^2}{\theta}\right)+\beta\left(\mu_0P_\rho\frac{\rho_x^2}{\rho}+\kappa\mu_0u_{xxx}\frac{\rho_x}{\rho}+\mu_0P_\theta\frac{\rho_x\theta_x}{\rho}-\mu_0\rho v_x^2\right)\right]\\&\quad=\gamma e^{\gamma t}\left[\mathcal{E}\left(\frac{1}{\rho},v,\eta,u_x\right)+\beta\left(\frac{1}{2}\mu_0^2\left(\frac{\rho_x}{\rho}\right)^2+\mu_0v\frac{\rho_x}{\rho}\right)\right]+e^{\gamma t}\left[\alpha\left(1-\frac{\overline{\theta}}{\theta}\right)\rho\theta_x\right.\\&\qquad\left.+(1-\beta)\mu_0\rho vv_x-\left(P-P\left(\frac{1}{\overline{\rho}},\overline{\theta}\right)\right)v+\kappa u_xv_x-\kappa u_{xx}v\right]_x,\end{aligned} \tag{3.3.101}$$

where $G(t)=e^{\gamma t}\left[\mathcal{E}\left(\frac{1}{\rho},v,\eta,u_x\right)+\beta\left(\frac{1}{2}\mu_0^2\left(\frac{\rho_x}{\rho}\right)^2+\mu_0v\frac{\rho_x}{\rho}\right)\right]$.

Multiplying (3.2.7) by v_{xx} in $L^2[0,\ 1]$, we have

$$\frac{d}{dt}\left(\frac{\kappa}{2}\|u_{xx}(t)\|^2+\frac{1}{2}\|v_x(t)\|^2\right)+\mu_0\int_0^1\frac{v_{xx}^2}{u}dx=\int_0^1\left(P_x+\mu_0\frac{u_xv_x}{u^2}\right)v_{xx}dx. \tag{3.3.102}$$

Integrating (3.3.101) over $[0,\ 1]$ with respect to x, multiplying (3.3.102) by $e^{\gamma t}$, adding the results up, and then using the Cauchy inequality and the Poincaré inequality, we derive from lemmas 3.3.11, 3.3.12 and 3.3.14–3.3.16 that for small $\beta>0$ and for any $\gamma>0$,

$$e^{\gamma t}\Big(\|\rho(t)-\overline{\rho}\|^2+\|v(t)\|^2+\|\eta(t)-\overline{\eta}\|^2+\|\rho_x(t)\|^2+\|v_x(t)\|^2+\|u_{xx}(t)\|^2\Big)$$
$$+\int_0^t e^{\gamma s}\Big(\|\rho_x\|^2+\|v_x\|^2+\|\theta_x\|^2+\|v_{xx}\|^2+\|u_{xx}\|^2\Big)(s)\,ds$$
$$\le C_1+C_1\gamma\int_0^t e^{\gamma s}\Big(\|\rho-\overline{\rho}\|^2+\|v\|^2+\big\|\theta-\overline{\theta}\big\|^2+\|\rho_x\|^2+\|v_x\|^2+\|u_{xx}\|^2\Big)(s)\,ds. \tag{3.3.103}$$

Therefore, we know from (3.3.103) that there exists a constant $\gamma_1'=\gamma_1'(C_1)>0$ such that for any fixed $\gamma\in(0,\,\gamma_1']$, (3.3.96) holds. □

Lemma 3.3.18. *Under assumptions of theorem 3.2.1, there exists a positive constant $\gamma_1=\gamma_1(C_1)\le\gamma_1'$ such that for any fixed $\gamma\in(0,\,\gamma_1]$, the generalized global solution $(u(t),\,v(t),\theta(t))$ in H_+^1 to the problem (3.2.6)–(3.2.12) satisfies for any $t\in(0,\,+\infty)$,*

$$e^{\gamma t}\Big(\|v_x(t)\|^2+\|\theta_x(t)\|^2\Big)+\int_0^t e^{\gamma s}\Big(\|v_{xx}\|^2+\|\theta_{xx}\|^2+\|u_{xxx}\|^2+\|v_t\|^2+\|\theta_t\|^2\Big)(s)\,ds\le C_1. \tag{3.3.104}$$

Proof. Noting the fact

$$\|u_{xxx}(t)\|\le C_1(\|u_x(t)\|+\|v_t(t)\|+\|\theta_x(t)\|+\|v_{xx}(t)\|),$$

and similarly to the proof of lemma 2.3.15, we can prove this lemma. □

Now using the above lemmas, we complete the proof of theorem 3.2.1. □

3.4 Global Existence and Exponential Stability in H_+^2

In this section, we shall complete the proof of theorem 3.2.2, that is, the global existence, asymptotic behavior and exponential stability of solutions to the problem (3.2.6)–(3.2.12) on H_+^2 under some relative assumptions. For the convenience, we suppose that assumptions in theorem 3.2.2 always hold in this section.

3.4.1 Global Existence and Uniqueness in H_+^2

We first deal with the boundedness of v_{xx} and θ_{xx}.

Lemma 3.4.1. *For any $t\in(0,\,+\infty)$, the following estimates hold*

$$\|v_t(t)\|^2+\|v_{xx}(t)\|^2+\int_0^t\|v_{xt}(s)\|^2\,ds\le C_2, \tag{3.4.1}$$

$$\int_0^t \|v_t(s)\|^2 ds \le C_2, \tag{3.4.2}$$

$$\|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2 + \int_0^t \|\theta_{xt}(s)\|^2 ds \le C_2. \tag{3.4.3}$$

Proof. Differentiating (3.2.7) with respect to t,

$$v_{tt} = -P_{xt} + \mu_0 \left(\frac{v_x}{u}\right)_{xt} - \kappa u_{xxxt}. \tag{3.4.4}$$

Multiplying (3.4.4) by v_t in Q_t, and using an integration by parts and the Cauchy inequality, we can derive the estimate (3.4.1) from corollary 3.3.3 and lemmas 3.3.10, 3.3.11.

Similarly, multiplying (3.2.7) by v_t in Q_t and using (3.4.1), we can get (3.4.2). The estimate (3.4.3) can be obtained by the same method as that in lemma 3.2 of Qin [103]. □

The next lemma shall concern the boundedness of u_{xxx}.

Lemma 3.4.2. *The following estimates hold for any $t \in (0, +\infty)$,*

$$\|u_{xxx}(t)\|^2 + \int_0^t \|u_{xxx}(s)\|^2 ds \le C_2, \tag{3.4.5}$$

$$\|u_{xxx}(t)\|^2 + \int_0^t \|v_{xxx}(s)\|^2 ds \le C_2, \tag{3.4.6}$$

$$\int_0^t \|\theta_{xxx}(s)\|^2 ds \le C_2. \tag{3.4.7}$$

Proof. The estimate (3.4.5) comes from the equation (3.2.7) and lemma 3.3.11. Multiplying the equation (3.2.7) by u_{xxxt}, noting that $u_{xxt} = v_{xxx}$, and using the Cauchy inequality, we have

$$\int_0^1 u_{xxx}^2 dx + \int_0^t \int_0^1 v_{xxx}^2 dx ds \le C \int_0^t \int_0^1 (v_{xt}^2 + u_{xx}^2 v_x^2 + u_x^2 v_{xx}^2 + v_x^2 u_x^4 + u_x^4 \\ + u_{xx}^2 + u_x^2 \theta_x^2 + \theta_x^4 + \theta_{xx}^2)\, dx ds,$$

which, combined with the Sobolev embedding theorem and lemmas 3.3.11 and 3.3.12, implies the estimate (3.4.6).

Differentiating (3.2.8) with respect to x, using the Young inequality, the Gagliardo–Nirenberg interpolation inequality and the Poincaré inequality, we can derive (3.4.7) from lemmas 3.3.1–3.3.12 and the estimate (3.4.3). □

Lemma 3.4.3. *As $t \to +\infty$, we have the following conclusions*

$$\|u(t) - \overline{u_0}\|_{H^3} \to 0,\ \|v(t)\|_{H^2} \to 0,\ \left\|\theta(t) - \overline{\theta}\right\|_{H^2} \to 0. \tag{3.4.8}$$

Proof. By the equation (3.2.7), we easily see that

$$\|u_{xxx}(t)\| \le C(\|v_t(t)\| + \|u_x(t)\| + \|\theta_x(t)\| + \|v_x(t)\| + \|v_{xx}(t)\|). \tag{3.4.9}$$

Multiplying (3.2.7) by u_{xxxt} in $L^2(0,\,1)$, we can derive from the estimate (3.4.3),

$$\frac{d}{dt}\|u_{xxx}(t)\|^2 \le C_1\Big(\|v_{tx}(t)\|^2 + \|u_{xx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|v_{xx}(t)\|^2 + 1\Big). \tag{3.4.10}$$

Differentiating (3.2.6) with respect to x and multiplying the resulting equation by v_{xxt}, we can obtain

$$\frac{d}{dt}\|v_{xx}(t)\|^2 \le C\Big(\|v_{xxx}(t)\|^2 + \|v_{xt}(t)\|^2\Big), \tag{3.4.11}$$

which, along with the Shen–Zheng inequality (theorem 1.2.10), leads to, as $t \to +\infty$,

$$\|v_{xx}(t)\| \to 0.$$

By the equation (3.2.8), we derive that

$$\|\theta_{xx}(t)\| \le C_2(\|\theta_t(t)\| + \|\theta_x(t)\| + \|v_{xx}(t)\| + \|u_x(t)\|). \tag{3.4.12}$$

Differentiating (3.3.5) with respect to t and multiplying the resultant by θ_t in $L^2(0,\,1)$, we have

$$\frac{d}{dt}\|\theta_t(t)\|^2 + \|\theta_{xt}(t)\|^2 \le C_2\Big(\|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|v_{xt}(t)\|^2\Big), \tag{3.4.13}$$

which, combined with lemmas 3.3.11 and the estimate (3.4.3), by the Shen–Zheng inequality (theorem 1.2.10), leads to, as $t \to +\infty$,

$$\|\theta_t(t)\| \to 0. \tag{3.4.14}$$

Furthermore, using theorem 3.2.1 and lemmas 3.3.11, 3.3.13 and 3.4.2, we can derive the desired estimate (3.4.8) from (3.4.10)–(3.4.12). □

Lemma 3.4.4. *The problem (3.2.6)–(3.2.12) admits a unique generalized global solution $(u(t),\, v(t),\, \theta(t))$ in H_+^2, which defines a nonlinear C_0-semigroup $S(t)$ (also denoted by the uniqueness of solutions in H_+^1) on H_+^2 such that for any initial data $(u_0, v_0, \theta_0) \in H_+^2$,*

$$\|S(t)(u_0, v_0, \theta_0)\|_{H_+^2} = \|(u(t), v(t), \theta(t))\|_{H_+^2} \le C_2,\ \ \forall t > 0, \tag{3.4.15}$$

$$S(t)(u_0, v_0, \theta_0) = (u(t), v(t), \theta(t)) \in C([0, +\infty), H_+^2), \tag{3.4.16}$$

$$u(t) \in C^{\frac{1}{2}}([0,+\infty), H^3), \quad v(t), \theta(t) \in C^{\frac{1}{2}}([0,+\infty), H^1). \tag{3.4.17}$$

Proof. The uniqueness of generalized global solutions in H_+^2 follows from that in H_+^1. Moreover, the proof of the continuity of C_0-semigroup $S(t)$ is also similar to that of lemma 3.3.14. See also the proof of lemma 3.4 in Qin [103].

For any $t_1 \geq 0$, $t > 0$, differentiating (3.2.6) with respect to x twice and integrating the resultant over $[t_1, t] \times [0, 1]$, and using the Hölder inequality, we can derive from lemmas 3.3.11 and 3.4.2 that

$$\|u(t) - u(t_1)\|_{H^2} \leq C_1 \left| \int_{t_1}^{t} \left(\|v_x\|^2 + \|v_{xx}\|^2 + \|v_{xxx}\|^2 \right)(s)\, ds \right|^{\frac{1}{2}} |t - t_1|^{\frac{1}{2}} \leq C_2 |t - t_1|^{\frac{1}{2}}.$$

Furthermore, by lemma 3.4.2, we also have

$$\|u(t) - u(t_1)\|_{H^3} \leq C_1 \left| \int_{t_1}^{t} \|u_{xxx}(s)\|^2 ds \right| + \|u(t) - u(t_1)\|_{H^2} \leq C_2 |t - t_1|^{\frac{1}{2}},$$

so

$$u(t) \in C^{\frac{1}{2}}([0,+\infty), H^3).$$

In the same manner we can show $v(t), \theta(t) \in C^{\frac{1}{2}}([0,+\infty), H^1)$. Thus we get (3.4.17). □

3.4.2 Exponential Stability in H_+^2

The following lemma gives us the exponential stability of semigroup $S(t)$ on H_+^2 (or, equivalently, of generalized global solution $(u(t), v(t), \theta(t))$ in H_+^2).

Lemma 3.4.5. *There exists a positive constant $\gamma_2 = \gamma_2(C_2) \leq \gamma_1$ such that for any fixed $\gamma \in (0, \gamma_2]$, the generalized global solution $(u(t), v(t), \theta(t))$ in H_+^2 to the problem (3.2.6)–(3.2.12) satisfies for any $t \in (0, +\infty)$,*

$$e^{\gamma t}\left(\|v(t)\|_{H^2}^2 + \|\theta(t) - \overline{\theta}\|_{H^2}^2 + \|u(t) - \overline{u}\|_{H^3}^2 \right) + \int_0^t e^{\gamma s}\left(\|v_{xt}\|^2 + \|\theta_{xt}\|^2 + \|v_{xxx}\|^2 \right)(s)\, ds \leq C_2. \tag{3.4.18}$$

Proof. Similarly to the proof of lemma 2.4.7, using the Sobolev embedding theorem and the Young inequality, we know from lemmas 3.3.11–3.3.18 that there exists a constant $\gamma_2 = \gamma_2(C_2) \leq \gamma_1$ such that for any fixed $\gamma \in (0, \gamma_2]$ and any $t \in (0, +\infty)$,

$$e^{\gamma t}\left(\|v_t(t)\|^2 + \|v_{xx}(t)\|^2 \right) + \int_0^t e^{\gamma s} \|v_{xt}(s)\|^2 ds \leq C_2. \tag{3.4.19}$$

Multiplying (3.2.7) by $u_{xxxt}e^{\gamma t}$ over Q_t, and using lemmas 3.3.11–3.3.18, the embedding theorem and the Young inequality, we easily derive

$$\begin{aligned}&\frac{1}{2}e^{\gamma t}\Big(\kappa\|u_{xxx}(t)\|^2+\|v_{xx}(t)\|^2\Big)+\mu_0\int_0^t e^{\gamma s}\Big\|\frac{v_{xxx}}{\sqrt{u}}\Big\|^2(s)\,ds\\&\quad\le C_2+\Big(C_2\gamma+\frac{\mu_0}{2}\Big)\int_0^t e^{\gamma s}\Big\|\frac{v_{xxx}}{\sqrt{u}}\Big\|^2(s)\,ds\\&\qquad+C_1\int_0^t e^{\gamma s}\Big(\|\theta_x\|^2+\|v_x\|^2+\|v_{xx}\|^2+\|u_{xx}\|^2+\|\theta_{xx}\|^2\Big)(s)\,ds.\end{aligned}$$

Similarly to the deduction of (3.4.19), we can also obtain

$$e^{\gamma t}\Big(\|u_{xxx}(t)\|^2+\|v_{xx}(t)\|^2\Big)+\int_0^t e^{\gamma s}\|v_{xxx}(s)\|^2\,ds\le C_2. \tag{3.4.20}$$

Analogously, we can easily derive from the equation (3.2.8) and lemmas 3.3.11–3.3.18 that

$$e^{\gamma t}\Big(\|\theta_t(t)\|^2+\|\theta_{xx}(t)\|^2\Big)+\int_0^t e^{\gamma s}\|\theta_{xt}(s)\|^2\,ds\le C_2, \tag{3.4.21}$$

which, combined with (3.4.19), (3.4.20) and lemmas 3.3.16, 3.3.17, yields the estimate (3.4.18). □

Now applying the above lemmas 3.4.1–3.4.5, we can complete the proof of theorem 3.2.2. □

3.5 Global Existence and Exponential Stability in H_+^4

In this section, we shall complete the proof of theorem 3.2.3, that is, the global existence, asymptotic behavior and exponential stability of solutions to the problem (3.2.6)–(3.2.12) under some relative assumptions. For the convenience, we also suppose that assumptions of theorem 3.2.3 hold in this section.

3.5.1 Global Existence in H_+^4

Firstly, we can obtain the following two lemmas which play an important role in our subsequent analysis.

Lemma 3.5.1. *The following estimate is valid for any $t\in(0,+\infty)$,*

$$\int_0^t\|u_x(s)\|_{H^3}^2\,ds\le C_4. \tag{3.5.1}$$

Proof. Differentiating (3.2.7) with respect to x, we have

$$v_{xt} = -P_{xx} + \mu_0 \left(\frac{v_x}{u}\right)_{xx} - \kappa u_{xxxx}, \tag{3.5.2}$$

then

$$\begin{aligned} \|u_{xxxx}(t)\| \le C(&\|v_{xt}\| + \|u_{xx}\| + \|\theta_{xx}\| + \left\|u_x^2\right\| + \|u_x\theta_x\| + \left\|\theta_x^2\right\| \\ &+ \|v_{xxx}\| + \|v_{xx}u_x\| + \|u_{xx}v_x\| + \left\|u_x^2 v_x\right\|)(t), \end{aligned}$$

which, together with lemmas 3.3.11, 3.3.12, 3.4.1 and 3.4.2, implies (3.5.1) by the embedding theorem. □

Lemma 3.5.2. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$ *and* $\varepsilon \in (0, 1)$ *small enough, we have the following estimate*

$$\begin{aligned} &\|v_x(t)\|_{H^2}^2 + \|u_x(t)\|_{H^3}^2 + \int_0^t \|v_x(s)\|_{H^3}^2 \, ds \\ &\quad \le C_4\varepsilon^{-4} + \varepsilon^2 \int_0^t \|v_{xxt}(s)\|^2 \, ds + \varepsilon \int_0^t \|u_{xxxt}(s)\|_{L^\infty}^2 \, ds. \end{aligned} \tag{3.5.3}$$

Proof. Multiplying (3.5.2) by u_{xxxxt} over Q_t, using lemmas 3.3.11, 3.3.12 and 3.4.1 and the embedding theorem, we get

$$\begin{aligned} &\|v_x(t)\|_{H^2}^2 + \|u_x(t)\|_{H^3}^2 + \int_0^t \|v_x(s)\|_{H^3}^2 \, ds \\ &\le C_4 + C_4 \int_0^t \left(\|v_{xxx}\|^2 + \|u_{xxx}\|^2\right)(s) \, ds \\ &\quad + \int_0^t \left(\|v_{xxx}u_{xxxt}\|_{L^\infty} + \|u_{xx}v_x u_{xxxt}\|_{L^\infty} + \|v_{xt}u_{xxxt}\|_{L^\infty}\right)(s) \, ds. \end{aligned} \tag{3.5.4}$$

By the embedding theorem and the Cauchy inequality, we can get for any $\varepsilon \in (0, 1)$ small enough,

$$\begin{aligned} \|v_{xxx}u_{xxxt}\|_{L^\infty} &\le C_4 \|v_{xxx}\|^{\frac{1}{2}} \|v_{xxxx}\|^{\frac{1}{2}} \|u_{xxxt}\|_{L^\infty} \\ &\le \varepsilon \|u_{xxxt}\|_{L^\infty}^2 + \varepsilon \|v_{xxxx}\|^2 + C_4\varepsilon^{-3} \|v_{xxx}\|^2, \end{aligned} \tag{3.5.5}$$

$$\|u_{xx}v_x u_{xxxt}\|_{L^\infty} \le \varepsilon \|u_{xxxt}\|_{L^\infty}^2 + C_4\varepsilon^{-1}\left(\|u_{xxx}\|^2 + \|u_{xx}\|^2\right), \tag{3.5.6}$$

$$\begin{aligned} \|v_{xt}u_{xxxt}\|_{L^\infty} &\le C_4 \|v_{xt}\|^{\frac{1}{2}} \|v_{xxt}\|^{\frac{1}{2}} \|u_{xxxt}\|_{L^\infty} \\ &\le \varepsilon \|u_{xxxt}\|_{L^\infty}^2 + \varepsilon^2 \|v_{xxt}\|^2 + C_4\varepsilon^{-4} \|v_{xt}\|^2. \end{aligned} \tag{3.5.7}$$

Inserting (3.5.5)–(3.5.7) into (3.5.4) implies the estimate (3.5.3) by lemmas 3.3.11, 3.4.1 and 3.4.2. □

Second, we establish the following estimates on the initial quantities.

Lemma 3.5.3. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$, *we have*

$$\|v_{xt}(x,0)\| + \|\theta_{xt}(x,0)\| \le C_4, \tag{3.5.8}$$

$$\|v_{tt}(x,0)\| + \|\theta_{tt}(x,0)\| + \|v_{xxt}(x,0)\| + \|\theta_{xxt}(x,0)\| \le C_4. \tag{3.5.9}$$

Proof. Similarly to the proofs of (2.5.3) and (2.5.4) in lemma 2.5.1, we easily complete the proof. Thus we omit the details. □

Third, we give some estimates on v_{tt}, θ_{tt}, v_{xt} and θ_{xt}.

Lemma 3.5.4. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$, *the following estimates hold*

$$\|v_{tt}(t)\|^2 + \|v_{xxt}(t)\|^2 + \int_0^t \|v_{xtt}(s)\|^2 ds \le C_4 + C_2 \int_0^t \|\theta_{xxt}(s)\|^2 ds, \tag{3.5.10}$$

$$\begin{aligned} &\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{xtt}(s)\|^2 ds \\ &\quad \le C_4\varepsilon^{-3} + C_2\varepsilon^{-1} \int_0^t \|\theta_{xxt}(s)\|^2 ds + C_1\varepsilon \int_0^t \left(\|v_{xxt}\|^2 + \|v_{xtt}\|^2\right)(s)\, ds. \end{aligned} \tag{3.5.11}$$

Proof. Differentiating (3.2.7) with respect to t twice, we have

$$v_{ttt} = -P_{xtt} + \mu_0 \left(\frac{v_x}{u}\right)_{xtt} - \kappa u_{xxxtt}. \tag{3.5.12}$$

Multiplying (3.5.12) by v_{tt} over Q_t, we can conclude from lemma 3.5.3 and the embedding theorem,

$$\begin{aligned} &\|v_{tt}(t)\|^2 + \|v_{xxt}(t)\|^2 + \int_0^t \|v_{xtt}(s)\|^2 ds \\ &\quad \le C_4 + C_1 \int_0^t \left(\|P_{tt}\|^2 + \|v_x v_{xt}\|^2 + \left\|v_x^3\right\|^2\right)(s)\, ds \\ &\quad \le C_4 + C_2 \int_0^t (\|v_x\|^2 + \|v_{xt}\|^2 + \|v_{xx}\|^2 + \|\theta_t\|^2 \\ &\qquad + \|\theta_{xt}\|^2 + \|\theta_{tt}\|^2)(s)\, ds. \end{aligned} \tag{3.5.13}$$

It follows from (3.2.8) that

$$\|\theta_{tt}(t)\| \le C_2\big(\|v_x(t)\|_{H^1} + \|u_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|v_{xt}(t)\| + \|\theta_{xxt}(t)\| + \|\theta_x(t)\|_{H^2}\big). \tag{3.5.14}$$

Inserting (3.5.14) to (3.5.13) and using lemmas 3.3.11, 3.3.12, 3.4.1 and 3.4.2, we obtain the estimate (3.5.10), while (3.5.11) can be obtained in a way similar to that in lemma 2.5.1. □

Lemma 3.5.5. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$*, we have the following estimates*

$$\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 ds \leq C_4, \tag{3.5.15}$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 ds \leq C_4\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t \left(\|v_{txx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{xxx}\|^2\|\theta_{xt}\|^2\right)(s)\, ds. \tag{3.5.16}$$

Proof. Differentiating (3.2.8) with respect to t, multiplying the resulting equation by v_{xxt} over Q_t, and performing an integration by parts and using the Cauchy inequality, we obtain

$$\begin{aligned}
&\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 ds \\
&\leq C_4 + C_0 \int_0^t \|v_x\|_{H^1}^2 + \|\theta_{xt}\|^2 + \|u_x v_x\|^2 + \|u_x\theta_t\|^2 + \|\theta_x v_x\|^2 \\
&\quad + \|\theta_x\theta_t\|^2 + \|v_{xx}v_x\|^2 + \|u_x v_{xt}\|^2 + \left\|u_x^2 v_x\right\|^2)(s)\, ds,
\end{aligned} \tag{3.5.17}$$

which, along with lemmas 3.3.11, 3.3.12 and 3.4.1–3.4.3, leads to the estimate (3.5.15) by the embedding theorem. The estimate (3.5.16) can be obtained in a way similar to that in lemma 2.5.2. □

Lemma 3.5.6. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$*, we have*

$$\begin{aligned}
&\|v_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|v_{xxt}(t)\|^2 + \|v_{xxx}(t)\|^2 \\
&\quad + \int_0^t \left(\|\theta_{xxt}\|^2 + \|\theta_{xtt}\|^2 + \|v_{xxt}\|^2 + \|v_{xtt}\|^2\right)(s)\, ds \leq C_4,
\end{aligned} \tag{3.5.18}$$

$$\|\theta_{xxt}(t)\| \leq C_4,\ \|\theta_{xxx}(t)\| \leq C_4. \tag{3.5.19}$$

Proof. Multiplying (3.5.10) and (3.5.11) by ε and ε^2, respectively; then adding the resultants to (3.5.16), choosing $\varepsilon \in (0, 1)$ small enough, we obtain

$$\begin{aligned}
&\|v_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|v_{xxt}(t)\|^2 + \int_0^t \left(\|\theta_{xxt}\|^2 + \|\theta_{xtt}\|^2 + \|v_{xtt}\|^2\right)(s)\, ds \\
&\leq C_4\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t \|\theta_{xxx}(s)\|^2\|\theta_{xt}(s)\|^2 ds,
\end{aligned}$$

which, along with the estimate (3.5.15) and by the Gronwall inequality, gives the estimate (3.5.18).

Differentiating (3.2.8) with respect to t, we derive from (3.5.18) that

$$\|\theta_{xxt}(t)\| \le C_2\|\theta_{tt}(t)\| + C_2(\|v_{xx}(t)\| + \|v_{xt}(t)\| + \|\theta_{xt}(t)\| + \|\theta_{xx}(t)\|) \le C_4.$$

Analogously, differentiating (3.2.8) with respect to x, we can complete the proof of (3.5.19) from the estimate (3.5.18). □

Finally, we obtain the global existence of solutions in H_+^4.

Lemma 3.5.7. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and any* $t \in (0, +\infty)$, *the following estimates hold*

$$\|u_x(t)\|_{H^3}^2 + \|u_x(t)\|_{W^{2,\infty}}^2 + \int_0^t \left(\|u_x\|_{H^3}^2 + \|u_x\|_{W^{2,\infty}}^2\right)(s)\,ds \le C_4, \tag{3.5.20}$$

$$\|v_x(t)\|_{H^3}^2 + \|v_x(t)\|_{H^2}^2 + \int_0^t \left(\|v_x\|_{H^3}^2 + \|v_t\|_{H^3}^2\right)(s)\,ds \le C_4, \tag{3.5.21}$$

$$\|\theta_x(t)\|_{H^3}^2 + \int_0^t \|\theta_x(s)\|_{H^3}^2\,ds \le C_4, \tag{3.5.22}$$

$$\int_0^t \left(\|v_{tt}\|^2 + \|\theta_{tt}\|^2\right)(s)\,ds \le C_4, \tag{3.5.23}$$

$$\|u_x(t)\|_{H^4}^2 + \int_0^t \left(\|u_x\|_{H^4}^2 + \|\theta_x\|_{H^4}^2 + \|\theta_t\|_{H^3}^2\right)(s)\,ds \le C_4, \tag{3.5.24}$$

$$\int_0^t \left(\|v_x\|_{H^4} + \|u_x\|_{H^5}\right)(s)\,ds \le C_4. \tag{3.5.25}$$

Proof. Differentiating (3.2.7) with respect to t, we have

$$v_{tt} = -P_{xt} + \mu_0\left(\frac{v_x}{u}\right)_{xt} - \kappa u_{xxxt}. \tag{3.5.26}$$

Multiplying (3.5.26) by v_{xxxxt} over Q_t, and using the embedding theorem and the Cauchy inequality, we conclude from the estimate (3.5.18),

$$\|v_t(t)\|_{H^2}^2 + \|v_x(t)\|_{H^3}^2 + \int_0^t \|v_{xxxt}(s)\|^2\,ds \le C_4. \tag{3.5.27}$$

We derive from the equation (3.5.26) that

$$\begin{aligned}\|u_{xxxt}\|_{L^\infty} \le C_2(&\|v_{tt}\|_{L^\infty} + \|v_{xx}\|_{L^\infty} + \|v_{xxt}\|_{L^\infty} + \|\theta_{xt}\|_{L^\infty} + \|v_x u_x\|_{L^\infty} + \|\theta_x u_x\|_{L^\infty}\\ &+ \|v_x\theta_x\|_{L^\infty} + \|\theta_x\theta_t\|_{L^\infty} + \|v_x v_{xx}\|_{L^\infty} + \|u_x v_{xt}\|_{L^\infty} + \left\|u_x v_x^2\right\|_{L^\infty}).\end{aligned} \tag{3.5.28}$$

Using the embedding theorem and the Cauchy inequality, we get from lemmas 3.3.11, 3.3.12 and 3.4.1–3.4.3,

$$\begin{aligned}\|u_{xxxt}(t)\|_{L^\infty}^2 \leq C_2(&\|v_{tt}(t)\|^2 + \|v_{xtt}(t)\|^2 + \|v_{xxt}(t)\|^2 + \|v_{xxx}(t)\|^2\\ &+ \|v_{xxxt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|v_{xx}(t)\|^2\\ &+ \|\theta_{xx}(t)\|^2 + \|v_{xt}(t)\|^2 + \|u_{xx}(t)\|^2).\end{aligned} \tag{3.5.29}$$

Thus, inserting (3.5.29) into (3.5.3), we obtain from the estimates (3.5.18) and (3.5.27),

$$\|v_x(t)\|_{H^2}^2 + \|u_x(t)\|_{H^3}^2 + \int_0^t \|v_x(s)\|_{H^3}^2\, ds \leq C_4, \tag{3.5.30}$$

which, along with lemma 3.5.1 and (3.5.27), implies the estimates (3.5.20) and (3.5.21).

Using lemma 3.5.6 and the embedding theorem, we easily get the estimates (3.5.22)–(3.5.24) from the estimate (3.5.30). Differentiating (3.2.7) with respect to x and t, we have

$$v_{xtt} = -P_{xxt} + \mu_0\left(\frac{v_x}{u}\right)_{xxt} - \kappa u_{xxxxt}, \tag{3.5.31}$$

thus, noting the fact that $u_{xxxxt} = v_{xxxxx}$, we can get

$$\begin{aligned}\|v_{xxxxx}(t)\| \leq C_4(&\|v_{xtt}(t)\| + \|v_{xxx}(t)\| + \|\theta_{xxt}(t)\| + \|\theta_{xt}(t)\| + \|v_{xt}(t)\|\\ &+ \|v_{xxxt}(t)\| + \|v_{xxt}(t)\| + \|u_{xx}(t)\|).\end{aligned} \tag{3.5.32}$$

Differentiating (3.2.7) with respect to x three times, we have

$$v_{xxxt} = -P_{xxxx} + \mu_0\left(\frac{v_x}{u}\right)_{xxxx} - \kappa u_{xxxxxx}, \tag{3.5.33}$$

thus,

$$\begin{aligned}\|u_{xxxxxx}(t)\| \leq C_4(&\|v_{xxxt}(t)\| + \|v_{xxxxx}(t)\| + \|\theta_{xxxx}(t)\| + \|\theta_{xxx}(t)\| + \|v_{xxx}(t)\|\\ &+ \|u_{xxxx}(t)\| + \|v_{xxxx}(t)\| + \|\theta_{xx}(t)\|),\end{aligned} \tag{3.5.34}$$

which, along with (3.5.32), leads to the desired estimate (3.5.25). □

Lemma 3.5.8. *For any* $(u_0, v_0, \theta_0) \in H_+^4$, *the following estimates are valid for any* $t \in (0, +\infty)$,

$$\begin{aligned}&\|u(t) - \overline{u_0}\|_{H^5}^2 + \|u_t\|_{H^4}^2 + \|u_t\|_{H^3}^2 + \|u_{tt}\|_{H^1}^2 + \|v(t)\|_{H^4}^2 + \|v_t\|_{H^2}^2 + \|v_{tt}\|^2\\ &\quad + \left\|\theta(t) - \overline{\theta}\right\|_{H^4}^2 + \|\theta_t\|_{H^2}^2 + \|\theta_{tt}\|^2 + \int_0^t (\|u - \overline{u_0}\|_{H^5}^2 + \|v\|_{H^5}^2 + \|v_t\|_{H^3}^2\\ &\quad + \|v_{tt}\|_{H^1}^2 + \left\|\theta - \overline{\theta}\right\|_{H^5}^2 + \|\theta_t\|_{H^3}^2 + \|\theta_{tt}\|_{H^1}^2)\, ds \leq C_4,\end{aligned} \tag{3.5.35}$$

$$\int_0^t \left(\|u_t\|_{H^4}^2 + \|u_{tt}\|_{H^2}^2 + \|u_{ttt}\|^2\right) ds \leq C_4. \tag{3.5.36}$$

Proof. Noting the equation (3.2.6), exploiting lemmas 3.3.11, 3.3.12, 3.4.1, 3.4.2, 3.5.1–3.5.7, we easily obtain the estimates (3.5.35) and (3.5.36). □

From lemmas 3.5.1–3.5.8, we have shown the global existence of solutions to the problem (3.2.6)–(3.2.12) in H_+^4 with the arbitrary initial data $(u_0, v_0, \theta_0) \in H_+^4$, the uniqueness of solutions in H_+^4 follows from that of solutions in H_+^1 or in H_+^2.

3.5.2 Asymptotic Behavior of Solutions in H_+^4

Now, we shall study the asymptotic behavior of solutions.

Lemma 3.5.9. *As $t \to +\infty$, there hold that*

$$\|u(t) - \overline{u_0}\|_{H^5} \to 0, \ \|v(t)\|_{H^4} \to 0, \ \left\|\theta(t) - \overline{\theta}\right\|_{H^4} \to 0. \tag{3.5.37}$$

Proof. Multiplying (3.5.2) by u_{xxxxt} in $L^2(0,1)$, using an integration by parts and the Sobolev embedding theorem, we can derive from the estimate (3.5.29),

$$\begin{aligned}\frac{d}{dt}\|u_{xxxx}(t)\|^2 + C_2^{-1}\|v_{xxxx}(t)\|^2 \leq C_4(&\|v_{xxxt}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|v_{xxx}(t)\|^2 \\ &+ \|v_{xxxx}(t)\|^2 + \|v_{tt}(t)\|^2 + \|v_{xtt}(t)\|^2 \\ &+ \|\theta_{xxx}(t)\|^2 + 1),\end{aligned} \tag{3.5.38}$$

which, combined with (3.5.25) and theorem 1.2.10 and lemma 3.5.6, gives us, as $t \to +\infty$,

$$\|u_{xxxx}(t)\| \to 0. \tag{3.5.39}$$

Differentiating (3.2.7) with respect to x twice, multiplying the resultant by u_{xxxxxt} in $L^2(0,1)$, and performing an integration by parts, we can derive by the embedding theorem,

$$\begin{aligned}\frac{d}{dt}\|u_{xxxxx}(t)\|^2 + \|v_{xxxxx}(t)\|^2 \leq C_4(&\|v_{xxxt}(t)\|^2 + \|u_{xxxx}(t)\|^2 + \|v_{xxxx}(t)\|^2 \\ &+ \|v_{xxx}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|\theta_{xxx}(t)\|^2 \\ &+ \|\theta_{xxxx}(t)\|^2 + 1),\end{aligned} \tag{3.5.40}$$

which, along with (3.5.39), lemma 3.5.8 and theorem 1.2.10, leads to, as $t \to +\infty$,

$$\|u_{xxxxx}(t)\| \to 0. \tag{3.5.41}$$

Thus, using the Poincaré inequality, we can get $\|u(t) - \overline{u_0}\|_{H^5} \to 0$.

Noting the fact $u_{xxt} = v_{xxx}$ from the equation (3.2.6), we have

$$\frac{d}{dt}\|v_{xxx}(t)\|^2 \leq C_4\Big(\|v_{xxx}(t)\|^2 + \|v_{xxxt}(t)\|^2\Big). \tag{3.5.42}$$

Multiplying (3.5.26) by v_{xxxxt} in $L^2(0, 1)$, we can obtain

$$\begin{aligned}\frac{d}{dt}\|v_{xxxx}(t)\|^2 + C_2^{-1}\|v_{xxxt}(t)\|^2 \leq{}& C_4(\|v_{xtt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|v_{xxt}(t)\|^2 + \|v_{xt}(t)\|^2\\ &+ \|\theta_{xxt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + 1),\end{aligned} \tag{3.5.43}$$

which, along with the estimate (3.5.42), lemma 3.5.8 and theorem 1.2.10, leads to, as $t \longrightarrow +\infty$,

$$\|v_{xxxx}(t)\| \to 0. \tag{3.5.44}$$

By the estimate (3.5.16), we can easily derive from the equation (3.2.8),

$$\begin{aligned}\frac{d}{dt}\|\theta_{xt}(t)\|^2 + C_2^{-1}\|\theta_{xxt}(t)\|^2 \leq{}& \varepsilon\Big(\|\theta_{xxxt}(t)\|^2 + \|v_{xxt}(t)\|^2\Big) + C_2(\|v_{xxx}(t)\|^2 + \|v_{xt}(t)\|^2\\ &+ \|\theta_{xxx}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|u_x(t)\|^2),\end{aligned} \tag{3.5.45}$$

which, combined with the Shen–Zheng inequality (theorem 1.2.10), leads to, as $t \longrightarrow +\infty$,

$$\|\theta_{xt}(t)\| \to 0. \tag{3.5.46}$$

We differentiate (3.2.8) with respect to x to derive

$$\|\theta_{xxx}(t)\| \leq C_2\big(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|\theta_x(t)\| + \|\theta_{xt}(t)\|\big), \tag{3.5.47}$$

which, along with the conclusion (3.5.46), as $t \longrightarrow +\infty$, we have

$$\|\theta_{xxx}(t)\| \to 0. \tag{3.5.48}$$

It follows from lemma 3.5.4 that

$$\begin{aligned}\frac{d}{dt}\|\theta_{tt}(t)\|^2 + C_2^{-1}\|\theta_{xxt}(t)\|^2 \leq{}& \varepsilon\Big(\|\theta_{xtt}(t)\|^2 + \|v_{xtt}(t)\|^2\Big) + C_2(\|v_{xx}(t)\|^2 + \|v_{xt}(t)\|^2\\ &+ \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2),\end{aligned} \tag{3.5.49}$$

which, by theorem 1.2.10, leads to, as $t \longrightarrow +\infty$,

$$\|\theta_{tt}(t)\| \to 0. \tag{3.5.50}$$

On the other hand, we have

$$\|\theta_{xxt}(t)\| \leq C_2(\|\theta_{tt}(t)\| + \|v_x(t)\| + \|u_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|\theta_{xxx}(t)\|),$$

which, combined with (3.4.13) and (3.5.46)–(3.5.50), implies, as $t \longrightarrow +\infty$,

$$\|\theta_{xxt}(t)\| \to 0. \tag{3.5.51}$$

Differentiating (3.2.8) with respect to x twice, we can derive

$$\|\theta_{xxxx}(t)\| \leq C_2\big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\|\big). \tag{3.5.52}$$

Thus it follows from (3.5.51) and (3.5.52) that, as $t \to +\infty$,

$$\|\theta_{xxxx}(t)\| \to 0, \tag{3.5.53}$$

which, along with the Shen–Zheng inequality (theorem 1.2.10) and the Poincaré inequality, leads to our desired estimate $\|\theta(t) - \overline{\theta}\|_{H^4} \to 0$, as $t \to +\infty$. □

In the following lemma, we shall establish the existence of a nonlinear C_0-semigroup $S(t)$ on H_+^4.

Lemma 3.5.10. *The global unique solution $(u(t), v(t), \theta(t))$ in H_+^4 to the problem (3.2.6)–(3.2.12) defines a nonlinear C_0-semigroup $S(t)$ on H_+^4 (also denoted by $S(t)$ by the uniqueness of solution in H_+^1 and H_+^2) such that for any initial data $(u_0, v_0, \theta_0) \in H_+^4$, there hold for any $t \in (0, +\infty)$,*

$$\|S(t)(u_0, v_0, \theta_0)\|_{H_+^4} = \|(u(t), v(t), \theta(t))\|_{H_+^4} \leq C_4, \tag{3.5.54}$$

$$S(t)(u_0, v_0, \theta_0) = (u(t), v(t), \theta(t)) \in C([0, +\infty), H_+^4). \tag{3.5.55}$$

Proof. Similarly to the proof of lemma 3.3.14 or lemma 3.4.4, we can complete the proof of (3.5.54) and (3.5.55). See also lemma 2.4.5 in Qin [99]. □

3.5.3 Exponential Stability in H_+^4

In the following lemmas, we shall make use of the estimates established in lemmas 3.5.1–3.5.9 to prove the exponential stability of global solutions or of the nonlinear C_0-semigroup $S(t)$ on H_+^4.

Lemma 3.5.11. *For any $(u_0, v_0, \theta_0) \in H_+^4$, there exists a positive constant $\gamma_4^{(1)} = \gamma_4^{(1)}(C_4) \leq \gamma_2$ such that for any fixed $\gamma \in \left(0, \gamma_4^{(1)}\right]$ and for any $t \in (0, +\infty)$ and $\varepsilon \in (0, 1)$ small enough, we have the following estimates,*

$$e^{\gamma t}\left(\|v_{tt}(t)\|^2 + \|v_{xxt}(t)\|^2\right) + \int_0^t e^{\gamma s}\|v_{ttx}(s)\|^2 ds \leq C_4, \tag{3.5.56}$$

$$\begin{aligned} &e^{\gamma t}\|\theta_{tt}(t)\|^2 + \int_0^t e^{\gamma s}\|\theta_{ttx}(s)\|^2 ds \\ &\quad \leq C_4\varepsilon^{-3} + C_4\varepsilon^{-1}\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2 ds + C_4\varepsilon\int_0^t e^{\gamma s}\left(\|v_{txx}\|^2 + \|v_{ttx}\|^2\right)(s)\, ds. \end{aligned} \tag{3.5.57}$$

Proof. Similarly to the proof of (3.5.10), we multiply (3.5.12) by $e^{\gamma t}v_{tt}$ in Q_t to obtain from lemmas 3.4.1 and 3.4.2,

$$\begin{aligned}\frac{1}{2}e^{\gamma t}\Big(\|v_{tt}(t)\|^2+\kappa\|v_{xxt}(t)\|^2\Big) &\le C_4-(C_1^{-1}-C_1\gamma)\int_0^t e^{\gamma s}\|v_{ttx}(s)\|^2ds+C_2\int_0^t e^{\gamma s}\|\theta_{tt}(s)\|^2ds\\ &\le C_4-(C_1^{-1}-C_1\gamma)\int_0^t e^{\gamma s}\|v_{ttx}(s)\|^2ds+C_4\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2ds\end{aligned}$$

which gives the estimate (3.5.56) by taking $0<\gamma\le\min\Big(\frac{1}{4C_1^2},\gamma_2\Big)$. And (3.5.57) can be also obtained, see lemma 2.5.4 for details. □

Lemma 3.5.12. *For any* $(u_0,v_0,\theta_0)\in H_+^4$, *there exists a positive constant* $\gamma_4^{(2)}=\gamma_4^{(2)}(C_4)\le\gamma_4^{(1)}$ *such that for any fixed* $\gamma\in\Big(0,\gamma_4^{(2)}\Big]$ *and for any* $t\in(0,+\infty)$ *and* $\varepsilon\in(0,1)$ *small enough, we have the following estimates,*

$$e^{\gamma t}\Big(\|v_{xt}(t)\|^2+\|v_{xxx}(t)\|^2\Big)+\int_0^t e^{\gamma s}\|v_{txx}(s)\|^2ds\le C_4, \tag{3.5.58}$$

$$e^{\gamma t}\|\theta_{xt}(t)\|^2+\int_0^t e^{\gamma s}\|\theta_{txx}(s)\|^2ds\le C_4\varepsilon^{-6}+C_2\varepsilon^2\int_0^t e^{\gamma s}\Big(\|v_{txx}\|^2+\|\theta_{ttx}\|^2\Big)(s)\,ds. \tag{3.5.59}$$

Proof. Similarly to the proof of (3.5.15), we multiply (3.5.26) by $e^{\gamma t}v_{xxt}$ in Q_t and use lemmas 3.4.1, 3.4.2, 3.4.5 and 3.3.18 to obtain

$$\frac{1}{2}e^{\gamma t}\Big(\|v_{xt}(t)\|^2+\kappa\|v_{xxx}(t)\|^2\Big)\le C_4-((2C_1)^{-1}-C_1\gamma)\int_0^t e^{\gamma s}\|v_{txx}(s)\|^2ds,$$

which gives (3.5.58) by taking $0<\gamma\le\min\Big(\frac{1}{8C_1^2},\gamma_4^{(1)}\Big)\equiv\gamma_4^{(2)}$. And (3.5.59) can be obtained, see lemma 2.5.5 for details. □

Lemma 3.5.13. *For any* $(u_0,v_0,\theta_0)\in H_+^4$, *there exists a positive constant* $\gamma_4\le\gamma_4^{(2)}$ *such that for any fixed* $\gamma\in(0,\gamma_4]$ *and for any* $t>0$, *the following estimates are valid,*

$$\begin{aligned}&e^{\gamma t}\Big(\|v_{tt}(t)\|^2+\|v_{xt}(t)\|^2+\|v_{xxt}(t)\|^2+\|v_{xxx}(t)\|^2+\|\theta_{xt}(t)\|^2+\|\theta_{tt}(t)\|^2\Big)\\ &\quad+\int_0^t e^{\gamma s}\Big(\|v_{txx}\|^2+\|v_{ttx}\|^2+\|\theta_{txx}\|^2+\|\theta_{ttx}\|^2\Big)(s)\,ds\le C_4,\end{aligned} \tag{3.5.60}$$

$$e^{\gamma t}\Big(\|u_{xxxx}(t)\|_{H^1}^2+\|u_{xxx}(t)\|_{W^{1,\infty}}^2\Big)+\int_0^t e^{\gamma s}\Big(\|u_{xxxx}\|_{H^1}^2+\|u_{xxx}\|_{W^{1,\infty}}^2\Big)(s)\,ds\le C_4, \tag{3.5.61}$$

$$\begin{aligned}
&e^{\gamma t}(\|v_{xxx}(t)\|_{H^1}^2+\|v_{xx}(t)\|_{W^{1,\infty}}^2+\|v_{xxt}(t)\|^2+\|\theta_{xx}(t)\|_{W^{1,\infty}}^2+\|\theta_{xxx}(t)\|_{H^1}^2\\
&\quad+\|\theta_{txx}(t)\|^2+\|u_{txxx}(t)\|^2)+\int_0^t e^{\gamma s}(\|v_{tt}\|^2+\|v_{xxxx}\|_{H^1}^2+\|v_{txx}\|_{H^1}^2\\
&\quad+\|v_{xx}\|_{W^{2,\infty}}^2+\|v_{tx}\|_{W^{1,\infty}}^2+\|\theta_{tt}\|^2+\|\theta_{xxxx}\|_{H^1}^2+\|\theta_{txx}\|_{H^1}^2\\
&\quad+\|\theta_{xx}\|_{W^{2,\infty}}^2+\|\theta_{tx}\|_{W^{1,\infty}}^2)(s)\,ds\le C_4.
\end{aligned}\tag{3.5.62}$$

Proof. Multiplying (3.5.56) and (3.5.57) by ε and $\varepsilon^{\frac{3}{2}}$, respectively, adding up the resultants, (3.5.58) and (3.5.59), and then taking $\varepsilon>0$ small enough, we can obtain the estimate (3.5.60). Similarly, we differentiate (3.2.7) with respect to x, multiply the resultant by $e^{\gamma t}u_{xxxxt}$ in Q_t and choose $\gamma>0$ so small that $0<\gamma\le\gamma_4=\min\left(\frac{1}{2C_1},\gamma_4^{(2)}\right)$, we conclude from lemmas 3.4.1, 3.4.2, 3.5.7 and 3.5.8 and the estimate (3.5.60) that for any $t\in(0,+\infty)$,

$$e^{\gamma t}\Big(\|u_{xxxx}(t)\|^2+\|v_{xxx}(t)\|^2\Big)+\int_0^t e^{\gamma s}\|v_{xxxx}(s)\|^2\,ds\le C_4.\tag{3.5.63}$$

Noting that the facts (3.5.47) and (3.5.52), we can derive from (3.5.18), (3.5.20)–(3.5.23), (3.5.35), (3.5.60) and (3.5.63) that for any fixed $\gamma\in(0,\gamma_4]$,

$$\begin{aligned}
&e^{\gamma t}(\|v_{xxx}(t)\|_{H^1}^2+\|v_{xx}(t)\|_{W^{1,\infty}}^2+\|v_{xxt}(t)\|^2+\|\theta_{xx}(t)\|_{W^{1,\infty}}^2+\|\theta_{xxx}(t)\|_{H^1}^2\\
&\quad+\|\theta_{txx}(t)\|^2)+\int_0^t e^{\gamma s}(\|v_{xxx}\|_{H^1}^2+\|v_{txx}\|^2+\|v_{xx}\|_{W^{1,\infty}}^2+\|\theta_{xxx}\|_{H^1}^2\\
&\quad+\|\theta_{txx}\|_{H^1}^2+\|\theta_{xx}\|_{W^{1,\infty}}^2)(s)\,ds\le C_4
\end{aligned}\tag{3.5.64}$$

and

$$\int_0^t e^{\gamma s}\Big(\|v_{tt}\|^2+\|v_{txxx}\|^2+\|\theta_{tt}\|^2+\|\theta_{txxx}\|^2\Big)(s)\,ds\le C_4.\tag{3.5.65}$$

Similarly to (3.5.63), we can also derive from the equation (3.2.7) that for any fixed $\gamma\in(0,\gamma_4]$ and any $t\in(0,+\infty)$,

$$e^{\gamma t}\|u_{xxxxx}(t)\|^2+\int_0^t e^{\gamma s}\|v_{xxxxx}(s)\|^2\,ds\le C_4.\tag{3.5.66}$$

Meanwhile, we also easily obtain that for any fixed $\gamma\in(0,\gamma_4]$,

$$\begin{aligned}
&\int_0^t e^{\gamma s}(\|u_{xxxxx}\|^2+\|v_{xxxxx}\|^2+\|\theta_{xxxxx}\|^2+\|u_{txxx}\|_{H^1}^2+\|v_{xx}\|_{W^{2,\infty}}^2\\
&\quad+\|\theta_{xx}\|_{W^{2,\infty}}^2+\|u_{xxxx}\|^2)(s)\,ds\le C_4,
\end{aligned}\tag{3.5.67}$$

which, combined with the estimates (3.5.60), (3.5.61)–(3.5.67) and the interpolation inequality, leads to the desired estimates (3.5.61) and (3.5.62). □

Now we can use the equation (3.2.6), lemmas 3.3.11, 3.3.12, 3.4.1, 3.4.2, theorems 3.2.1, 3.2.2 and lemmas 3.5.1–3.5.13 to prove the following lemma.

Lemma 3.5.14. *For any* $(u_0, v_0, \theta_0) \in H_+^4$ *and for any fixed* $\gamma \in (0, \gamma_4]$ *and any* $t \in (0, +\infty)$*, the following estimate holds,*

$$\begin{aligned} &e^{\gamma t}(\|u(t)-\overline{u}\|_{H^5}^2 + \|v(t)\|_{H^4}^2 + \left\|\theta(t)-\overline{\theta}\right\|_{H^4}^2 + \|u_t(t)\|_{H^3}^2 \\ &\quad + \|u_{tt}(t)\|_{H^1}^2 + \|v_t(t)\|_{H^2}^2 + \|v_{tt}(t)\|^2 + \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2) \\ &\quad + \int_0^t e^{\gamma s}(\|u-\overline{u}\|_{H^5}^2 + \|v\|_{H^5}^2 + \left\|\theta-\overline{\theta}\right\|_{H^5}^2 + \|v_t\|_{H^3}^2 + \|v_{tt}\|_{H^1}^2 + \|\theta_t\|_{H^3}^2 \\ &\quad + \|\theta_{tt}\|_{H^1}^2 + \|u_t\|_{H^4}^2 + \|u_{tt}\|_{H^2}^2 + \|u_{ttt}\|^2)(s)\,ds \le C_4. \qquad (3.5.68) \end{aligned}$$

Now using lemmas 3.5.1–3.5.14, we can complete the proof of theorem 3.2.3. □

3.6 Bibliographic Comments

A century ago, Korteweg [74] first introduced the formulation of the theory of capillarity with diffuse interfaces and Dunn and Serrin [37] actually derived using the second gradient theory (see also [61, 132]). Here the main difficulty consists in describing the location and the movement of the interfaces.

There are some known results concerning the Korteweg type system. In the corresponding isothermal model, Danchin and Desjardins [19] studied the well-posedness of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, they proved the existence of solutions close to a stable equilibrium, and local in time existence for solutions when the pressure law may present spinodal regions. Bresch *et al.* [5] considered a global existence of weak solutions in dimension 2 or 3 using the regularity of the density in a periodic domain $\Omega = T^n$. Subsequently, they discussed the case of a smooth bounded domain depending on the choice of the diffusion term and on the boundary conditions, with a positive result in the case $\Omega = (0, 1) \times T^{n-1}$. Haspot [50] improved the existence of global weak solutions in dimension 2 for initial data in the energy space, close to a stable equilibrium and with specific choices on the capillary coefficients. Kotschote [75] proved existence and uniqueness of local strong solutions for this system by Dore-Venni theory, real interpolation, $\mathcal{H}^\infty$-calculus and the contraction mapping principle. Tan, Wang and Xu [128] proved the global existence of a strong solution with the small initial perturbation in $\mathbb{R}^3$. In the non-isothermal model, Kotschote [76] addressed the existence and uniqueness of strong solutions on bounded domain with compact boundaries by refined methods of maximal regularity. Recently, Kotschote [77] established global existence and uniqueness of strong solutions for initial data near equilibria and showed exponential stability of equilibria in the phase space.

However, most of these results are concentrated on the high dimensional case and, to the best of our knowledge, fewer results have been obtained for the one-dimensional case. Charve and Haspot [11] only studied the one-dimensional isothermal case and proved the existence of a global strong solution, then showed the global strong solution of the Korteweg system converges in the case of a γ law for the pressure ($P(\rho) = a\rho^{\gamma}$, $\gamma > 1$) to a weak-entropy solution of the compressible Euler equations. Haspot [50] showed the existence of global weak solutions for a specific type of capillary coefficients with large initial data in the energy space. With the other specific type of capillary coefficients $\kappa = \rho^{-5}$, Tsyganov [133] proved the global existence and asymptotic convergence of weak solutions on the Lagrangian coordinates by a easier method different from those in Kato [70] and Bona and Sachs [6]. Moreover, Tsyganov [133] studied the model in the finite interval $[0, 1]$. Tan and Guo [127] extended the existence theorem in [133] and considered a problem extending the finite region to the half space $\mathbb{R}_+$. Recently, Chen [12, 14] investigated the non-isothermal problem and considered the time-asymptotic behavior of solutions to one-dimensional compressible fluid models of Korteweg type, *i.e.*, strong rarefaction waves and the viscous contact waves, which govern the motions of the compressible fluids with internal capillarity.

Chapter 4

Exponential Stability of Solutions for the Compressible p-th Power Newtonian Fluid with Large Initial Data

4.1 Introduction

It is well known that the study on gas dynamical model is a hot topic in many fields, such as fluid dynamics, physics, fluid mechanics and applied mathematics.

As we know, the state of the ideal gas is determined by its density ρ, pressure P and absolute temperature θ and is interrelated by the law associated with the names of Boyle, Gay–Lussac and Mariotte

$$P = R\rho\theta$$

where R is the universal gas constant. In addition, barotropic thermodynamic processes with no regard to temperature may be treated in the realm of mechanics, such as $P = P(\rho)$, especially, the simplest example is an isothermal process, that is, in which the temperature is constant, so that

$$P = R\rho.$$

Subtler is the case of an isentropic or adiabatic process, in which the temperature and the specific volume vary simultaneously in such a proportion that the amount of heat stored in any part of the gas remains fixed. As shown by Laplace and Poisson, this assumption leads to

$$P = R\rho^p$$

where $p > 1$ is a constant and called the adiabatic exponent.

The p-th power Newtonian fluid is a linearly viscous, Fourier heat-conducting compressible gas, whose pressure P and internal energy e are given by

$$P = R\rho^p\theta, \quad e = c_v\theta$$

with the pressure exponent $p \geq 1$ and constant specific heat $c_v > 0$.

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4.2 Model and Main Results

In this book, we shall establish the exponential stability of global solutions in H^i $(i = 1,\ 2,\ 4)$ for the compressible Navier–Stokes equations which describe the motion of the p-th power Newtonian fluid. We assume that the corresponding solutions only depend on the radial variable r and the time variable $t \in [0, +\infty)$. Here we use $\overrightarrow{\mathbf{U}}(\mathbf{x},\ t)$ to denote the velocity of fluid and $\mathbf{\Omega} = \{\mathbf{x} |\ 0 < a \leq |\mathbf{x}| \leq b < +\infty\}$ is a symmetric domain in $\mathbb{R}^N$. As in [18, 44, 53], the equations can be read as

$$\begin{cases} \rho_t + (\rho u)_r + \frac{m\rho u}{r} = 0, \\ \rho\left(u_t + uu_r - \frac{v^2}{r}\right) + P_r = \beta\left(u_r + \frac{mu}{r}\right)_r, \\ \rho\left(v_t + uv_r + \frac{uv}{r}\right) = \mu\left(v_r + \frac{mv}{r}\right)_r, \\ \rho(w_t + uw_r) = \mu\left(w_{rr} + \frac{mw_r}{r}\right), \\ c_v\rho(\theta_t + u\theta_r) + P\left(u_r + \frac{mu}{r}\right) = k\left(\theta_{rr} + \frac{m\theta_r}{r}\right) + Q, \end{cases} \tag{4.2.1}$$

where the viscosity coefficients μ, λ satisfy the natural restrictions $\mu > 0$, $N\lambda + 2\mu \geq 0$ and the coefficient of heat conduction k is positive $k > 0$, $\beta = 2\mu + \lambda$ and

$$Q = \lambda\left(u_r + \frac{mu}{r}\right)^2 + \mu\left\{\left(v_r - \frac{mv}{r}\right)^2 + w_r^2 + 2u_r^2 + \frac{2mu^2}{r^2}\right\}.$$

In the spherically symmetric case, $m = N - 1,\ r = |\mathbf{x},\ \overrightarrow{\mathbf{U}}(\mathbf{x},\ t) = u(r, t)\frac{\mathbf{x}}{r}$, $v = w = 0$. In the cylindrically symmetric case, $m = 1,\ r = \sqrt{x_1^2 + x_2^2}$, $\overrightarrow{\mathbf{U}}(\mathbf{x}, t) = u(r, t)\frac{(x_1, x_2, 0)}{r} + v(r, t)\frac{(-x_2, x_1, 0)}{r} + w(r, t)(0,\ 0,\ 1)$. The following boundary and initial conditions can be given by

$$(u, v, w, \theta_r)(a, t) = (u, v, w, \theta_r)(b, t) = 0, \quad t \in [0, +\infty), \tag{4.2.2}$$

$$(\rho, u, v, w, \theta)(r, 0) = (\rho_0, u_0, v_0, w_0, \theta_0)(r), \quad r \in [a, b]. \tag{4.2.3}$$

As in [18, 65, 103], we denote by $\eta = \frac{1}{\rho}$ the specific volume of the flows and can transfer the problem (4.2.1)–(4.2.3) into the equations in Lagrangian coordinates as follows

$$\left\{\begin{aligned} &\eta_t = (r^m u)_x, \qquad 0 < x < L, t > 0, && (4.2.4) \\ &u_t = r^m\left(-\frac{R\theta}{\eta^p} + \beta\frac{(r^m u)_x}{\eta}\right)_x + \frac{v^2}{r}, && (4.2.5) \\ &v_t = \mu r^m\left(\frac{(r^m v)_x}{\eta}\right)_x - \frac{uv}{r}, && (4.2.6) \\ &w_t = \mu r^m\left(\frac{(r^m w)_x}{\eta}\right)_x + \frac{\mu m\eta w}{r^2}, && (4.2.7) \\ &c_v\theta_t = \left(-\frac{R\theta}{\eta^p} + \beta\frac{(r^m u)_x}{\eta}\right)(r^m u)_x + \left(k\frac{r^{2m}\theta_x}{\eta}\right)_x + \mu\frac{(r^m v)_x^2}{\eta} \\ &\qquad\quad + \mu\frac{r^{2m}w_x^2}{\eta} - 2\mu m(r^{m-1}u^2 + r^{m-1}v^2)_x. && (4.2.8) \end{aligned}\right.$$

Here $\eta(x,\ t)$, $u(x,\ t)$, $v(x,\ t)$, $w(x,\ t)$ and $\theta(x,\ t)$ are the specific volume, radial velocity, angular velocity, axial velocity and absolute temperature of the fluid, respectively. Moreover, radius r also depends on the Lagrangian mass coordinates $(x,\ t)$, $L = \int_a^b y^m \rho(y,t)\,dy = \int_a^b y^m \rho_0\,dy$ and

$$Q = \frac{\beta(r^m u)_x^2 + \mu(r^m v)_x^2 + \mu r^{2m} w_x^2}{\eta^2} - \frac{2\mu m(r^{m-1}u^2 + r^{m-1}v^2)_x}{\eta},$$

whereas boundary and initial conditions are

$$(u, v, w, \theta_x)(0,t) = (u, v, w, \theta_x)(L,t) = 0, \tag{4.2.9}$$

$$(\eta, u, v, w, \theta)(x,0) = (\eta_0, u_0, v_0, w_0, \theta_0)(x). \tag{4.2.10}$$

It is well known that

$$r_t = u, \quad r_x = \frac{\eta}{r^m}, \tag{4.2.11}$$

and

$$r(0,t) = a, \quad r(L,t) = b, \quad r_0(x) = \left(a^{m+1} + (m+1)\int_0^x \eta_0(y)\,dy\right)^{\frac{1}{m+1}}. \tag{4.2.12}$$

In what follows, we use letters C_i $(i = 1,\ 2,\ 4)$ to denote the generic positive constants depending on the norms of initial data $(u_0,\ v_0,\ w_0,\ \theta_0)$ in H^i but being independent of t.

We are now in a position to state our main theorems.

Theorem 4.2.1. *Assume that the initial data* $(\eta_0, u_0, v_0, w_0, \theta_0) \in (H^1[0,\ L])^5$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\eta,\ u,\ v,\ w,\ \theta)$ $\in L^\infty([0,\ +\infty),\ (H^1[0,\ L])^5)$ *to problem* (4.2.4)–(4.2.10) *verifying for all* $(x,\ t)$ $\in [0,\ L] \times [0,\ +\infty)$,

$$0 < C_1^{-1} \le \eta(x,t) \le C_1, \quad 0 < C_1^{-1} \le \theta(x,t) \le C_1, \tag{4.2.13}$$

and

$$\begin{aligned} &\|\eta(t) - \overline{\eta}\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^1}^2 \\ &\quad + \|r(t) - \overline{r}\|_{H^2}^2 + \|r_t(t)\|_{H^1}^2 + \int_0^t \Big(\|\eta - \overline{\eta}\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|w\|_{H^2}^2 \\ &\quad + \left\|\theta - \overline{\theta}\right\|_{H^2}^2 + \|r - \overline{r}\|_{H^2}^2 + \|r_t\|_{H^2}^2 + \|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \|\theta_t\|^2\Big)(s)\,ds \le C_1. \end{aligned} \tag{4.2.14}$$

Moreover, there exist constants $C_1 > 0$ *and* $\gamma_1 = \gamma_1(C_1) > 0$ *such that for any fixed* $\gamma \in (0,\ \gamma_1]$, *the following estimate holds for any* $t > 0$,

$$\begin{aligned}
&e^{\gamma t}\Big(\|\eta(t)-\overline{\eta}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\|v(t)\|_{H^1}^2+\|w(t)\|_{H^1}^2+\big\|\theta(t)-\overline{\theta}\big\|_{H^1}^2\\
&\quad+\|r(t)-\overline{r}\|_{H^2}^2+\|r_t(t)\|_{H^1}^2\Big)+\int_0^t e^{\gamma s}\Big(\|\eta-\overline{\eta}\|_{H^1}^2+\|u\|_{H^2}^2+\|v\|_{H^2}^2+\|w\|_{H^2}^2\\
&\quad+\big\|\theta-\overline{\theta}\big\|_{H^2}^2+\|r-\overline{r}\|_{H^2}^2+\|r_t\|_{H^2}^2+\|u_t\|^2+\|v_t\|^2+\|w_t\|^2+\|\theta_t\|^2\Big)(s)\,ds\le C_1
\end{aligned}\tag{4.2.15}$$

where $\overline{\eta}=\overline{\eta_0}=L^{-1}\int_0^L\eta_0(x)\,dx$, *constant* $\overline{\theta}>0$ *is determined by* $c_v\overline{\theta}=\int_0^L\left(\frac{1}{2}(u_0^2+v_0^2+w_0^2)+c_v\theta_0\right)dx$ *and* $\overline{r}=(a^2+2\overline{\eta}x)^{\frac{1}{2}}$.

Theorem 4.2.2. *Assume that the initial data* $(\eta_0,u_0,v_0,w_0,\theta_0)\in(H^2[0,\,L])^5$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\eta,\,u,\,v,\,w,\,\theta)\in L^\infty([0,+\infty),\ (H^2[0,\ L])^5)$ *to problem (4.2.4)–(4.2.10) verifying for all* $(x,\,t)\in[0,\,L]\times[0,+\infty)$,

$$\begin{aligned}
&\|\eta(t)-\overline{\eta}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|v(t)\|_{H^2}^2+\|w(t)\|_{H^2}^2+\big\|\theta(t)-\overline{\theta}\big\|_{H^2}^2+\|r(t)-\overline{r}\|_{H^3}^2\\
&\quad+\|r_t(t)\|_{H^2}^2+\int_0^t\Big(\|\eta-\overline{\eta}\|_{H^2}^2+\|u\|_{H^3}^2+\|v\|_{H^3}^2+\|w\|_{H^3}^2+\big\|\theta-\overline{\theta}\big\|_{H^3}^2\\
&\quad+\|r-\overline{r}\|_{H^3}^2+\|r_t\|_{H^3}^2+\|\eta_t\|_{H^2}^2+\|u_{xt}\|^2+\|v_{xt}\|^2+\|w_{xt}\|^2+\|\theta_{xt}\|^2)(s)\,ds\le C_2.
\end{aligned}\tag{4.2.16}$$

Moreover, there exist constants $C_2>0$ *and* $\gamma_2=\gamma_2(C_2)>0$ *such that for any fixed* $\gamma\in(0,\,\gamma_2]$, *the following estimate holds for any* $t>0$,

$$\begin{aligned}
&e^{\gamma t}\Big(\|\eta(t)-\overline{\eta}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|v(t)\|_{H^2}^2+\|w(t)\|_{H^2}^2+\big\|\theta(t)-\overline{\theta}\big\|_{H^2}^2+\|r(t)-\overline{r}\|_{H^3}^2\\
&\quad+\|r_t(t)\|_{H^2}^2\Big)+\int_0^t e^{\gamma s}\Big(\|\eta-\overline{\eta}\|_{H^2}^2+\|u\|_{H^3}^2+\|v\|_{H^3}^2+\|w\|_{H^3}^2+\big\|\theta-\overline{\theta}\big\|_{H^3}^2\\
&\quad+\|r-\overline{r}\|_{H^3}^2+\|r_t\|_{H^3}^2+\|\eta_t\|_{H^2}^2+\|u_{xt}\|^2+\|v_{xt}\|^2+\|w_{xt}\|^2+\|\theta_{xt}\|^2\Big)(s)\,ds\le C_2
\end{aligned}\tag{4.2.17}$$

where $\overline{\eta},\overline{\theta}$ *and* $\overline{r}$ *are the same as those in theorem 4.2.1.*

Theorem 4.2.3. *Assume that the initial data* $(\eta_0,u_0,v_0,w_0,\theta_0)\in(H^4[0,\,L])^5$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\eta,\,u,\,v,\,w,\,\theta)\in L^\infty([0,\ +\infty),\ (H^4[0,\,L])^5)$ *to problem (4.2.4)–(4.2.10) verifying for all* $(x,\,t)\in[0,\,L]\times[0,+\infty)$,

$$\begin{aligned}
&\|\eta(t)-\overline{\eta}\|_{H^4}^2+\|u(t)\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|w(t)\|_{H^4}^2+\big\|\theta(t)-\overline{\theta}\big\|_{H^4}^2+\|r(t)-\overline{r}\|_{H^5}^2\\
&\quad+\|r_t(t)\|_{H^4}^2+\int_0^t\Big(\|\eta-\overline{\eta}\|_{H^4}^2+\|u\|_{H^5}^2+\|v\|_{H^5}^2+\|w\|_{H^5}^2+\big\|\theta-\overline{\theta}\big\|_{H^5}^2\\
&\quad+\|r-\overline{r}\|_{H^5}^2+\|r_t\|_{H^5}^2+\|u_t\|_{H^3}^2+\|v_t\|_{H^3}^2+\|w_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2)(s)\,ds\le C_4.
\end{aligned}\tag{4.2.18}$$

Moreover, there exist constants $C_4>0$ *and* $\gamma_4=\gamma_4(C_4)>0$ *such that for any fixed* $\gamma\in(0,\,\gamma_4]$, *the following estimate holds for any* $t>0$,

$$
\begin{aligned}
&e^{\gamma t}(\|\eta(t)-\overline{\eta}\|_{H^4}^2+\|u(t)\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|w(t)\|_{H^4}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^4}^2+\|r(t)-\overline{r}\|_{H^5}^2\\
&\quad+\|r_t(t)\|_{H^4}^2\Big)+\int_0^t e^{\gamma s}\Big(\|\eta-\overline{\eta}\|_{H^4}^2+\|u\|_{H^5}^2+\|v\|_{H^5}^2+\|w\|_{H^5}^2+\left\|\theta-\overline{\theta}\right\|_{H^5}^2\\
&\quad+\|r-\overline{r}\|_{H^5}^2+\|r_t\|_{H^5}^2+\|u_t\|_{H^3}^2+\|v_t\|_{H^3}^2+\|w_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2\Big)(s)\,ds\le C_4 \qquad (4.2.19)
\end{aligned}
$$

where $\overline{\eta},\overline{\theta}$ *and* $\overline{r}$ *are the same as those in theorem 4.2.1.*

Remark 4.2.1. *On the one hand, for the spherically symmetric case, our theorems not only generalize the results of [65] where the author considered the case* $p=1$, *but also we obtain global existence and exponential stability of global solutions in* H^i $(i=2,\ 4)$. *On the other hand, for the cylindrically symmetric case, we have generalized and improved the results in [99, 103, 111, 114] by removing the smallness condition of total energy* $E_0=\int_0^L[\frac{1}{2}(u_0^2+v_0^2+w_0^2)+c_v\theta_0]\,dx$.

In this chapter, we remedy a defect in [18] and further establish exponential stability of the global spherically and cylindrically symmetric solutions in H^i $(i=1,\ 2,\ 4)$ for the p-th power compressible Newtonian fluid with large initial data. These results of this chapter come from [141].

4.3 Global Existence and Exponential Stability in H^1

In this section, we shall show theorem 4.2.1, that is, we shall establish the global existence and exponential stability for the generalized solutions in H^1 to problem (4.2.4)–(4.2.10) in terms of a series of lemmas.

4.3.1 Global Existence in H^1

First, we can obtain the following lemma by equations.

Lemma 4.3.1. *Under assumptions in theorem 4.2.1, there exists a constant* $C_1>0$ *such that the following estimates hold,*

$$\int_0^L \eta(x,t)\,dx=\int_0^L \eta_0(x)\,dx=\eta^*=\frac{b^{m+1}-a^{m+1}}{m+1}, \qquad (4.3.1)$$

$$\int_0^L (u^2+v^2+w^2)(x,t)\,dx+\int_0^t\int_0^L\left(\frac{\eta Q}{\theta}+k\frac{r^{2m}\theta_x^2}{\eta\theta^2}+\frac{r^{2m}u_x^2}{\eta\theta}+\frac{w_x^2}{\eta\theta}\right)dxds\le C_1, \qquad (4.3.2)$$

$$0<C_1^{-1}\le\int_0^L \theta(x,t)\,dx\le C_1. \qquad (4.3.3)$$

Proof. Integrating (4.2.4) over $x \in [0, L]$ and using the initial and boundary conditions (4.2.9)–(4.2.10) yield (4.3.1). Now we take

$$h(\eta) = \begin{cases} \log \eta, & \text{if } p = 1, \\ \dfrac{1}{p-1}(\overline{\eta}^{1-p} - \eta^{1-p}), & \text{if } p > 1, \end{cases}$$

and

$$\varphi(\theta) = \theta - \log\theta + 1, \qquad \psi(\eta) = \eta - h(\eta) + 1.$$

Also, we denote

$$\sigma = -\frac{R\theta}{\eta^p} + \beta\frac{(r^m u)_x}{\eta}, \quad q = -k\frac{r^{2m}\theta_x}{\eta}.$$

Then it follows from (4.2.4)–(4.2.8) that

$$\begin{aligned} &\left(\frac{1}{2}(u^2+v^2+w^2) + c_v\varphi(\theta) + R\psi(\eta)\right)_t + \frac{\eta Q}{\theta} + \frac{kr^{2m}\theta_x^2}{\eta\theta^2} \\ &\quad = \Bigg(r^m u\sigma + \mu\frac{r^m v(r^m v)_x}{\eta} + \mu\frac{r^{2m}ww_x}{\eta} - 2\mu m(r^{m-1}u^2 + r^{m-1}v^2) \\ &\qquad - \left(1-\frac{1}{\theta}\right)q + Rr^m u\Bigg)_x. \end{aligned} \tag{4.3.4}$$

Integrating (4.3.4) over $[0, L] \times [0, t]$ and using the boundary conditions (4.2.9), we obtain

$$\int_0^L \left(\frac{1}{2}(u^2+v^2+w^2) + c_v\varphi(\theta) + R\psi(\eta)\right)dx + \int_0^t\int_0^L \left(\frac{\eta Q}{\theta} + \frac{kr^{2m}\theta_x^2}{\eta\theta^2}\right)dxds \le C_1. \tag{4.3.5}$$

Now noting that (4.2.11), we have

$$\begin{aligned} &\frac{\beta}{\eta}(r^m u)_x^2 - 2\mu m(r^{m-1}u^2)_x \\ &\quad = \frac{2\mu(2\mu+(m+1)\lambda)}{2\mu+m\lambda}\frac{r^{2m}u_x^2}{\eta} + \frac{m(2\mu+m\lambda)}{\eta}\left(\frac{\eta u}{r} + \frac{\lambda r^m u_x}{2\mu+m\lambda}\right)^2 \\ &\quad \ge \frac{2\mu(2\mu+(m+1)\lambda)}{2\mu+m\lambda}\frac{r^{2m}u_x^2}{\eta} \end{aligned} \tag{4.3.6}$$

and

$$\frac{(r^m v)_x^2}{\eta} - 2m(r^{m-1}v^2)_x = \frac{1}{\eta}\left(r^m v_x - \frac{m\eta v}{r}\right)^2 \ge 0 \tag{4.3.7}$$

where we have used the fact that $v = 0$, if $m \geq 2$. Therefore, with the help of (4.3.5)–(4.3.7), we obtain (4.3.2). By the Jensen inequality, we easily obtain (4.3.3) from (4.3.5). This proof mainly comes from Cui and Yao [18]. □

Lemma 4.3.2. *Under assumptions in theorem 4.2.1, the following estimates hold*

$$\int_0^t \int_0^L v^2\, dxds \leq C_1, \tag{4.3.8}$$

$$C_1^{-1} - C_1 V(t)\eta_{\max} \leq \theta(x, t) \leq C_1 + C_1 V(t)\eta_{\max} \tag{4.3.9}$$

with $V(t) = \int_0^L \frac{r^{2m}\theta_x^2}{\eta\theta^2} dx$ *and* $\eta_{\max} = \max_{x\in[0, L]}\eta(x, t)$.

Proof. Using (4.2.11), we can rewrite (4.2.6) as

$$(rv)_t = \mu r^{m+1}\left(\frac{(r^m v)_x}{\eta}\right)_x. \tag{4.3.10}$$

Multiplying (4.3.10) by rv, integrating the result with respect to x over $[0, L]$ and using (4.2.9) and (4.2.11), we have

$$\begin{aligned}\left(\frac{1}{2}\int_0^L |rv|^2\, dx\right)_t &= -\mu\int_0^L (r^{m+2}v)_x \frac{(r^m v)_x}{\eta}\, dx \\ &= -\mu\int_0^L \frac{r^2 (r^m v)_x^2}{\eta} dx - \mu\int_0^L \frac{[(r^m v)^2]_x}{r^{m-1}} dx \\ &= -\mu\int_0^L \frac{r^2 (r^m v)_x^2}{\eta} dx,\end{aligned} \tag{4.3.11}$$

where we have used the fact that $v = 0$, if $m \geq 2$.

Integrating (4.3.11) with respect to t over $[0, t]$ to get

$$\int_0^t \int_0^L \frac{r^2 (r^m v)_x^2}{\eta} dxds \leq C_1$$

which, along with (4.3.1), the Poincaré inequality and the Hölder inequality, yields

$$\begin{aligned}\int_0^t \int_0^L v^2\, dxds &\leq C_1 \int_0^t \max_{x\in[0, L]} |r^m v|^2\, ds \\ &\leq C_1 \int_0^t \left(\int_0^L |(r^m v)_x|^2\, dx\right)^2 ds \\ &\leq C_1 \int_0^t \left(\int_0^L \eta\, dx\right)\left(\int_0^L \frac{(r^m v)_x^2}{\eta} dx\right) ds \\ &\leq C_1 \int_0^t \int_0^L \frac{r^2 (r^m v)_x^2}{\eta} dxds \\ &\leq C_1.\end{aligned}$$

It follows from (4.3.3) that there exists a point $x_0(t) \in [0,\, L]$, by means of the mean value theorem, such that

$$0 \leq C_1^{-1} \leq \theta(x_0(t), t) = \int_0^L \theta(x, t)\, dx \leq C_1. \tag{4.3.12}$$

By Sobolev's embedding theorem, we have

$$|\theta^{\frac{1}{2}}(x, t) - \theta^{\frac{1}{2}}(x_0(t), t)| \leq C_1 \Big| \int_{x_0(t)}^x \frac{\theta_x}{\sqrt{\theta}}(y, t)\, dy \Big| \leq C_1 \int_0^L \frac{|\theta_x|}{\sqrt{\theta}}(x, t)\, dx. \tag{4.3.13}$$

By the Hölder inequality, we obtain from (4.3.13) that

$$\begin{aligned} \theta(x, t) &\leq C_1 + C_1 \left(\int_0^L \frac{|\theta_x|}{\sqrt{\theta}}(x, t)\, dx \right)^2 \\ &\leq C_1 + C_1 \left(\int_0^L \frac{r^{2m} |\theta_x|^2}{\eta \theta^2} dx \right) \left(\int_0^L \frac{\eta \theta}{r^{2m}} dx \right) \\ &\leq C_1 + C_1 \eta_{\max}(t) \int_0^L \frac{r^{2m} \theta_x^2}{\eta \theta^2} dx. \end{aligned} \tag{4.3.14}$$

In the same manner, we also obtain from (4.3.13) that

$$\begin{aligned} \theta(x, t) &\geq C_1 - C_1 \left(\int_0^L \frac{|\theta_x|}{\sqrt{\theta}}(x, t)\, dx \right)^2 \\ &\geq C_1 - C_1 \left(\int_0^L \frac{r^{2m} |\theta_x|^2}{\eta \theta^2} dx \right) \left(\int_0^L \frac{\eta \theta}{r^{2m}} dx \right) \\ &\geq C_1 - C_1 \eta_{\max}(t) \int_0^L \frac{r^{2m} \theta_x^2}{\eta \theta^2} dx, \end{aligned} \tag{4.3.15}$$

which, together with (4.3.14), yields the estimate (4.3.9). □

The next lemma deals with the expression of the specific volume η and comes from lemma 3.4 in Cui and Yao [18].

Lemma 4.3.3. *Under assumptions in theorem 4.2.1, there exists a point $x_1(t) \in [0,\, L]$ such that*

$$\eta^p(x, t) = D(x, t) \left(\frac{1}{A(x, t)} + \frac{Rp}{\beta} \int_0^t \frac{\theta(x, s) A(x, s)}{D(x, s) A(x, t)} ds \right) \tag{4.3.16}$$

where

$$\begin{aligned} D(x, t) &= \eta_0^p(x) \exp\left(\frac{p}{\beta \eta^*} \int_0^L \eta_0 \int_0^x \frac{u_0}{r_0^m}(y)\, dy dx + \frac{p}{\beta} \int_{x_1(t)}^x \left(\frac{u}{r^m} - \frac{u_0}{r_0^m} \right) dy - \frac{p}{\beta} \int_0^{x_1(t)} \frac{u_0}{r_0^m} dy \right), \\ A(x, t) &= \exp\left(\frac{p}{\beta \eta^*} \int_0^t \int_0^L \left(\frac{u^2 + v^2}{m+1} + \frac{R\theta}{\eta^{p-1}} \right) dyds + \frac{p a^{m+1}}{\beta (m+1) \eta^*} \int_0^t \int_0^L \frac{m u^2 - v^2}{r^{m+1}} dyds \right. \\ &\qquad \left. + \frac{p}{\beta} \int_0^t \int_x^L \frac{m u^2 - v^2}{r^{m+1}} dyds \right). \end{aligned}$$

Proof. Define the similar auxiliary functions as in [65, 99]

$$\Lambda(x,t)=\frac{\beta(r^m u)_x}{\eta}-\frac{R\theta}{\eta^p}+\int_x^L\frac{mu^2-v^2}{r^{m+1}}dy$$

and

$$\phi(x,t)=\int_0^t\Lambda(x,s)\,ds+\int_0^x\frac{u_0}{r_0^m}dy.$$

Then we easily get

$$\phi_x=\frac{u}{r^m} \tag{4.3.17}$$

and

$$\phi_t=\Lambda=\frac{\beta(r^m u)_x}{\eta}-\frac{R\theta}{\eta^p}+\int_x^L\frac{mu^2-v^2}{r^{m+1}}dy. \tag{4.3.18}$$

Multiplying (4.3.18) by η, using (4.2.4), (4.2.11) and (4.3.17), we can know

$$\begin{aligned}(\phi\eta)_t-(\phi r^m u)_x&=-u^2-\frac{R\theta}{\eta^{p-1}}+\beta(r^m u)_x+\frac{1}{m+1}(r^{m+1})_x\int_x^L\frac{mu^2-v^2}{r^{m+1}}dy\\&=-\frac{u^2+v^2}{m+1}-\frac{R\theta}{\eta^{p-1}}+\beta(r^m u)_x+\frac{1}{m+1}\left(r^{m+1}\int_x^L\frac{mu^2-v^2}{r^{m+1}}dy\right)_x.\end{aligned} \tag{4.3.19}$$

Integrating (4.3.19) over $[0,L]\times[0,t]$, using the boundary conditions (4.2.9) and (4.2.12), we have

$$\begin{aligned}\int_0^L(\eta\phi)(x,t)\,dx=&\int_0^L(\eta_0\phi_0)(x)\,dx-\int_0^t\int_0^L\left(\frac{u^2+v^2}{m+1}+\frac{R\theta}{\eta^{p-1}}\right)dxds\\&-\frac{a^{m+1}}{m+1}\int_0^t\int_0^L\frac{mu^2-v^2}{r^{m+1}}dxds.\end{aligned} \tag{4.3.20}$$

On the other hand, by virtue of (4.3.1) and the mean value theorem, we know for each $t\geq0$ there is an $x_1(t)\in[0,L]$ such that

$$\eta^*\phi(x_1(t),t)=\int_0^L(\phi\eta)(x,t)\,dx,$$

which, together with (4.3.20), implies, for any $t\geq0$,

$$\begin{aligned}\int_0^t\Lambda(x_1(t),s)\,ds&=\phi(x_1(t),t)-\int_0^{x_1(t)}\frac{u_0}{r_0^m}dy\\&=\frac{1}{\eta^*}\left(\int_0^L(\eta_0\phi_0)(x)\,dx-\int_0^t\int_0^L\left(\frac{u^2+v^2}{m+1}+\frac{R\theta}{\eta^{p-1}}\right)dxds\right.\\&\quad\left.-\frac{a^{m+1}}{m+1}\int_0^t\int_0^L\frac{mu^2-v^2}{r^{m+1}}dxds\right)-\int_0^{x_1(t)}\frac{u_0}{r_0^m}dy.\end{aligned} \tag{4.3.21}$$

By means of (4.2.4) and (4.2.11), we can rewrite (4.2.5) as

$$\left(\frac{u}{r^m}\right)_t = \beta(\log\eta)_{xt} - \left(\frac{R\theta}{\eta^p}\right)_x + \frac{v^2 - mu^2}{r^{m+1}} = \Lambda_x. \tag{4.3.22}$$

Integrating (4.3.22) over $[0, t]$ with respect to t, then integrating the result over $[x_1(t), x]$ with respect to x and using (4.3.21), we have

$$\begin{aligned}
&\beta\log\eta(x,t) - \int_0^t \frac{R\theta}{\eta^p}ds\\
&\quad= \beta\log\eta_0(x) + \int_0^t \Lambda(x_1(t), s)\,ds - \int_0^t\int_x^L \frac{mu^2 - v^2}{r^{m+1}}dxds\\
&\qquad + \int_{x_1(t)}^x \left(\frac{u}{r^m} - \frac{u_0}{r_0^m}\right)dy\\
&\quad= \beta\log\eta_0(x) + \frac{1}{\eta^*}\left(\int_0^L \eta_0\phi_0\,dx - \int_0^t\int_0^L \left(\frac{u^2+v^2}{m+1} + \frac{R\theta}{\eta^{p-1}}\right)dxds\right.\\
&\qquad\left. - \frac{a^{m+1}}{m+1}\int_0^t\int_0^L \frac{mu^2 - v^2}{r^{m+1}}dxds\right) - \int_0^{x_1(t)} \frac{u_0}{r_0^m}dy\\
&\qquad - \int_0^t\int_x^L \frac{mu^2 - v^2}{r^{m+1}}dxds + \int_{x_1(t)}^x \left(\frac{u}{r^m} - \frac{u_0}{r_0^m}\right)dy.
\end{aligned} \tag{4.3.23}$$

Multiplying (4.3.23) by $\frac{p}{\beta}$, and taking the exponentials to the result, we know

$$\frac{d}{dt}\exp\left(\frac{Rp}{\beta}\int_0^t \frac{\theta}{\eta^p}ds\right) = \frac{Rp\theta(x,t)A(x,t)}{\beta D(x,t)}. \tag{4.3.24}$$

Integrating (4.3.24) over $[0, t]$ with respect to t yields (4.3.16). $\square$

The next lemma is important to get the lower upper of the specific volume whose proof can be found in [79]. Here we omit it.

Lemma 4.3.4. *Let* $a(t) \geq 0$, $q(t) \in C[0, +\infty)$. *If there exist constants* A_i $(i = 1, 2, 3, 4)$ *such that for any* $0 \leq s \leq t$,

$$A_1 e^{A_2(t-s)} \leq \exp\left(\int_s^t q(\xi)\,d\xi\right) \leq A_3 e^{A_4(t-s)},$$

then we have

$$\begin{aligned}
C_1^{-1}\liminf_{t\to+\infty}\int_t^{t+1} a(s)\,ds &\leq \liminf_{t\to+\infty}\int_0^t \exp\left(-\int_s^t q(\xi)\,d\xi\right)a(s)\,ds\\
&\leq \limsup_{t\to+\infty}\int_0^t \exp\left(-\int_s^t q(\xi)\,d\xi\right)a(s)\,ds\\
&\leq C_1\limsup_{t\to+\infty}\int_t^{t+1} a(s)\,ds.
\end{aligned}$$

Proof. Here we only give the estimate from above, and the estimate from below is derived similarly. In fact, we have, for some $T > 0$,

$$\begin{aligned}
&\int_0^t \exp\Big(-\int_s^t q(\xi)\,d\xi\Big)a(s)\,ds\\
&\quad\le \exp\Big(-\int_0^t q(\xi)\,d\xi\Big)\int_0^{T+1}\exp\Big(\int_0^s q(\xi)\,d\xi\Big)a(s)\,ds\\
&\qquad+\sum_{j=0}^{\max\{0,[t-T]-1\}}\int_{t-j-1}^{t-j}\exp\Big(-\int_0^s q(\xi)\,d\xi\Big)a(s)\,ds\\
&\quad\le C(T)\Big(-\int_0^t q(\xi)\,d\xi\Big)+A_1^{-1}\Big(\sup_{t\ge T}\int_t^{t+1}a(s)\,ds\Big)\sum_{j=0}^{\max\{0,[t-T]-1\}}e^{-A_2 j}.
\end{aligned}$$

Here $[\cdot]$ is a Gaussian symbol. Therefore the desired estimate follows. □

The next lemma concerns the upper and lower bounds of the specific volume and removes the smallness of initial data. Here, we remedy a defect in [18].

Lemma 4.3.5. *Under assumptions in theorem 4.2.1, the following estimate holds, for all* $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \le \eta(x,t) \le C_1. \tag{4.3.25}$$

Proof. Let

$$\begin{aligned}
M(x,t) &= \frac{p}{\beta\eta^*}\int_0^L\Big(\frac{u^2+v^2}{m+1}+\frac{R\theta}{\eta^{p-1}}\Big)dy+\frac{pa^{m+1}}{\beta(m+1)\eta^*}\int_0^L\frac{mu^2-v^2}{r^{m+1}}dy\\
&\quad+\frac{p}{\beta}\int_x^L\frac{mu^2-v^2}{r^{m+1}}dy.
\end{aligned}\tag{4.3.26}$$

Then we have

$$\frac{A(x,s)}{A(x,t)} = \exp\Big(-\int_s^t M(x,\tau)\,d\tau\Big). \tag{4.3.27}$$

By lemma 4.3.1, we easily know

$$M(x,t) \ge C_1\int_0^L \frac{\theta}{\eta^{p-1}}dx - C_1\int_0^L v^2\,dx. \tag{4.3.28}$$

Using (4.3.12) and this inequality $\log x \le x - 1 \le x$, for any $x > 0$, we have for all $t > s$,

$$\begin{aligned}
\exp\Big(\frac{1}{t-s}\int_s^t \log\theta(x_0(\tau),\tau)\,d\tau\Big) &\ge \exp\Big(-\frac{1}{t-s}\int_s^t|\log\theta(x_0(\tau),\tau)|\,d\tau\Big)\\
&\ge \exp\big(\log C_1^{-1} - C_1(t-s)\big)
\end{aligned}\tag{4.3.29}$$

where $x_0(t)$ is defined in (4.3.12). On the other hand, for any fixed s, t satisfying $0 \le s \le t$, we have, by the Young inequality,

$$\begin{aligned}\int_s^t \int_0^x \left|\frac{\theta_x(y,\tau)}{\theta(y,\tau)}\right| dyd\tau &\le \frac{1}{2}\int_s^t \int_0^x \frac{\theta_x^2}{\theta^2\eta} dyd\tau + \frac{1}{2}\int_s^t \int_0^x \eta\, dyd\tau \\ &\le C_1 + C_1(t-s), \end{aligned} \tag{4.3.30}$$

which, together with the convexity of e^x and (4.3.11), leads to

$$\begin{aligned}\int_s^t \theta(x,\tau)\,d\tau &= \int_s^t \exp\{\log\theta(x,\tau)\}\,d\tau \\ &\ge (t-s)\exp\left(\frac{1}{t-s}\int_s^t \log\theta(x,\tau)\,d\tau\right) \\ &= (t-s)\exp\left(\frac{1}{t-s}\int_s^t \left(\int_{x_0(\tau)}^x \frac{\theta_x}{\theta}dy + \log\theta(x_0(\tau),\tau)\right)d\tau\right) \\ &\ge (t-s)\exp\left(\log C_1 - \frac{C_1}{t-s} - \frac{1}{t-s}\int_s^t \int_0^x \left|\frac{\theta_x}{\theta}\right| dyd\tau\right) \\ &\ge C_1^{-1}(t-s)\exp\left(-\frac{C_1}{t-s}\right). \end{aligned} \tag{4.3.31}$$

Therefore, taking infimum in x to both sides of (4.3.31), we have

$$\int_s^t \inf_{x\in[0,\,L]} \theta(x,\tau)\,d\tau \ge \frac{t-s}{C_1} - C_1. \tag{4.3.32}$$

Now, by virtue of the convexity of $x^{-(p-1)}$ and the Jensen inequality, we derive from lemma 4.3.1 for any $0 \le s \le t$,

$$\begin{aligned}\int_s^t \int_0^L \frac{\theta}{\eta^{p-1}} dxd\tau &\ge \int_s^t \inf_{x\in[0,\,L]} \theta(x,\tau)\left(\int_0^L \eta^{-(p-1)}\,dx\right)d\tau \\ &\ge C_1 \int_s^t \inf_{x\in[0,\,L]} \theta(x,\tau)\left(\int_0^L \eta\,dx\right)^{-(p-1)} d\tau \\ &\ge \frac{t-s}{C_1} - C_1 \end{aligned} \tag{4.3.33}$$

which, together with lemma 4.3.2, leads to

$$\begin{aligned}-\int_s^t M(x,\tau)\,d\tau &\le -C_1\int_s^t \int_0^L \frac{\theta}{\eta^{p-1}} dyd\tau + C_1\int_s^t \int_0^L v^2\,dyd\tau \\ &\le C_1 - \frac{t-s}{C_1}. \end{aligned} \tag{4.3.34}$$

Thus it follows from (4.3.9) and (4.3.16) that

$$\frac{A(x,s)}{A(x,t)} \le C_1 \exp\left(-\frac{t-s}{C_1}\right) \tag{4.3.35}$$

which, along with the expression of $A(x,t)$, implies (*i.e.*, $s = 0$)

$$\frac{1}{A(x,t)} \leq C_1 \exp\left(-\frac{t}{C_1}\right) \leq C_1. \tag{4.3.36}$$

By virtue of lemma 4.3.1, we easily get, for all $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \leq D(x,t) \leq C_1 \tag{4.3.37}$$

which, combined with (4.3.1), (4.3.5), (4.3.6), (4.3.17) and (4.3.18), yields that

$$\begin{aligned}\eta^p(x,t) &\leq C_1 + C_1\int_0^t e^{-\frac{t-s}{C_1}}\theta(x,s)\,ds\\ &\leq C_1 + C_1\int_0^t e^{-\frac{t-s}{C_1}}ds + C_1\int_0^t \eta_{\max}(s)\,V(s)\,ds\\ &\leq C_1 + C_1\int_0^t \eta_{\max}^p(s)\,V(s)\,ds\end{aligned}$$

where we have used the Young inequality, for all $p \geq 1$,

$$\eta_{\max}(s) \leq C_1 + C_1\eta_{\max}^p(s).$$

Thus, we have

$$\eta_{\max}^p(t) \leq C_1 + C_1\int_0^t \eta_{\max}^p(s)\,V(s)\,ds$$

which, by the Gronwall inequality, implies

$$\eta_{\max}^p(t) \leq C_1. \tag{4.3.38}$$

In what follows, we deal with the lower bound of the specific volume η (see also lemma 3.6 in Cui and Yao [18]).

We can rewrite (4.2.5) or (4.3.22) in the following form

$$\begin{aligned}\left(\frac{u}{r^m}\right)_t &= \beta(\log\eta)_{xt} - \left(\frac{R\theta}{\eta^p}\right)_x + \frac{v^2 - mu^2}{r^{m+1}}\\ &= \sigma_x + \frac{v^2 - mu^2}{r^{m+1}}.\end{aligned} \tag{4.3.39}$$

Integrating (4.3.39) over $[\tau, t] \times [x, L]$ for any fixed τ and t satisfying $0 \leq \tau \leq t$, we have, for any $0 \leq x \leq L$,

$$\begin{aligned}&\int_L^x \left(\frac{u}{r^m}(y,t) - \frac{u}{r^m}(y,\tau)\right)dy + \int_\tau^t\int_L^x \frac{mu^2 - v^2}{r^{m+1}}dyds\\ &= \int_\tau^t \sigma(x,s)\,ds - \int_\tau^t \sigma(L,s)\,ds\\ &= -\int_\tau^t \frac{R\theta}{\eta^p}ds + \beta\log\eta(x,t) - \beta\log\eta(x,\tau) - \int_\tau^t \sigma(L,s)\,ds.\end{aligned} \tag{4.3.40}$$

Multiplying (4.3.40) by $\frac{p}{\beta}$, taking the exponentials to the result, we can get

$$\frac{d}{dt}\left(\eta^p(x,t)e^{-\frac{p}{\beta}(I(\tau,t)+B(x;\tau,t))}\right)=\frac{Rp}{\beta}\theta(x,t)e^{-\frac{p}{\beta}(I(\tau,t)+B(x;\tau,t))} \tag{4.3.41}$$

where

$$I(\tau,t)=\int_\tau^t\sigma(L,s)\,ds$$

and

$$B(x;\tau,t)=\int_L^x\left(\frac{u}{r^m}(y,t)-\frac{u}{r^m}(y,\tau)\right)dy+\int_\tau^t\int_L^x\frac{mu^2-v^2}{r^{m+1}}dyds.$$

Thus it follows from (4.3.41) that

$$\eta^p(x,t)e^{-\frac{p}{\beta}(I(\tau,t)+B(x;\tau,t))}=\eta^p(x,\tau)+\int_\tau^t\frac{Rp}{\beta}\theta(x,s)e^{-\frac{p}{\beta}(I(\tau,s)+B(x;\tau,s))}ds. \tag{4.3.42}$$

In addition, using (4.3.2), (4.3.3) and (4.3.38), the Poincaré inequality and the Hölder inequality, we have

$$\begin{aligned}\int_0^t\int_0^L\frac{mu^2}{r^{m+1}}dyds&\le C_1\int_0^t\max_{x\in[0,L]}|u|^2ds\\&\le C_1\int_0^t\left(\int_0^L|u_x|dx\right)^2ds\\&\le C_1\int_0^t\left(\int_0^L\theta\,dx\right)\left(\int_0^L\frac{|u_x|^2}{\theta}dx\right)ds\\&\le C_1\int_0^t\int_0^L\frac{|u_x|^2}{\eta\theta}dxds\\&\le C_1,\end{aligned} \tag{4.3.43}$$

which, together with (4.3.2) and (4.3.8), yields

$$|B(x;\tau,t)|\le C_1. \tag{4.3.44}$$

Combining (4.3.42) and (4.3.44) leads to

$$\begin{aligned}C_1^{-1}\left(\eta^p(x,\tau)+\int_\tau^t\theta(x,s)e^{-\frac{p}{\beta}I(\tau,s)}ds\right)&\le\eta^p(x,t)e^{-\frac{p}{\beta}I(\tau,t)}\\&\le C_1\left(\eta^p(x,\tau)+\int_\tau^t\theta(x,s)e^{-\frac{p}{\beta}I(\tau,s)}ds\right).\end{aligned} \tag{4.3.45}$$

On the other hand, by the Hölder inequality, we derive from (4.3.1) that

$$\int_0^L\eta^p\,dx\ge C_1^{-1}\left(\int_0^L\eta\,dx\right)^p\ge C_1^{-1}. \tag{4.3.46}$$

Meanwhile, we know from (4.3.38) that

$$\int_0^L \eta^p \, dx \le C_1. \tag{4.3.47}$$

Inserting (4.3.45) with respect to x over $[0,\ L]$ and using (4.3.46), (4.3.47) and (4.3.3), we have

$$C_1^{-1}\left(1+\int_\tau^t e^{-\frac{p}{\beta}I(\tau,s)}\,ds\right) \le e^{-\frac{p}{\beta}I(\tau,t)} \le C_1\left(1+\int_\tau^t e^{-\frac{p}{\beta}I(\tau,s)}\,ds\right). \tag{4.3.48}$$

Then, by the Gronwall inequality, we derive

$$C_1^{-1} e^{C_1^{-1}(t-\tau)} \le e^{-\frac{p}{\beta}I(\tau,t)} \le C_1 e^{C_1(t-\tau)}. \tag{4.3.49}$$

Setting $\tau = 0$ in the left-hand side in (4.3.45) while utilizing (4.3.9), (4.3.48) and (4.3.49), we obtain

$$\begin{aligned}
\eta^p(x,t) &\ge C_1^{-1} e^{\frac{p}{\beta}I(0,t)}\left(\eta^p(x,0)+\int_0^t \theta(x,s)e^{-\frac{p}{\beta}I(0,s)}\,ds\right)\\
&\ge C_1^{-1} e^{\frac{p}{\beta}I(0,t)}\left(1+\int_0^t e^{-\frac{p}{\beta}I(0,s)}\,ds - C_1\int_0^t e^{-\frac{p}{\beta}I(0,s)}\,V(s)\eta_{\max}\,ds\right)\\
&\ge C_1^{-1} - C_1\int_0^t e^{\frac{p}{\beta}I(s,t)}\,V(s)\,ds.
\end{aligned} \tag{4.3.50}$$

Furthermore, by virtue of lemma 4.3.4 with $q(t) = -\frac{p}{\beta}\sigma(L,t)$ and $a(t) = V(t)$, we know

$$\lim_{t\to+\infty}\int_0^t e^{\frac{p}{\beta}I(s,t)}\,V(s)\,ds = 0, \tag{4.3.51}$$

where we have used (4.3.2) and (4.3.49). Therefore, there exists $\tilde{T}$ satisfying

$$\eta^p(x,t) \ge C_1^{-1}, \quad \text{if } t \ge \tilde{T}. \tag{4.3.52}$$

Finally, if $0 \le t \le \tilde{T}$, then we obtain from the left-hand side of (4.3.45) and (4.3.49) that

$$\begin{aligned}
\eta^p(x,t) &\ge C_1^{-1} e^{\frac{p}{\beta}I(0,t)}\left(\eta^p(x,0)+\int_0^t \theta(x,s)e^{-\frac{p}{\beta}I(0,s)}\,ds\right)\\
&\ge C_1^{-1} e^{-C_1^{-1}t} \ge C_1^{-1} e^{-C_1^{-1}\tilde{T}} \ge C_1^{-1},
\end{aligned} \tag{4.3.53}$$

which, along with (4.3.52), leads to, for any $t \ge 0$,

$$\eta(x,t) \ge C_1^{-1}.$$

Thus we complete the proof of the lemma. □

It should be noted that if we get the uniform upper and lower bounds of the specific volume, the exponentially asymptotical behavior could be obtained by the classical energy method as in [65, 79, 105]. For completeness, we still present some crucial estimates mainly from [18].

Lemma 4.3.6. *Under assumptions in theorem 4.2.1, the following estimates hold for any* $t > 0$,

$$\|v(t)\|_{L^\infty}^2 + \|v_x(t)\|^2 + \int_0^t \Big(\|v_t\|^2 + \|v\|_{H^1}^2 + \|v\|_{L^\infty}^2\Big)(s)\,ds \le C_1, \tag{4.3.54}$$

$$\|w(t)\|_{L^\infty}^2 + \|w_x(t)\|^2 + \int_0^t \Big(\|w_t\|^2 + \|w\|_{H^1}^2 + \|w\|_{L^\infty}^2\Big)(s)\,ds \le C_1. \tag{4.3.55}$$

Proof. Multiplying (4.2.6) by v, integrating the resulting equation with respect to x over $[0, L]$ and using the Poincaré inequality, (4.2.9) and (4.2.11), we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L v^2\,dx + \mu\int_0^L \frac{(r^m v)_x^2}{\eta}dx &\le C_1\int_0^L |u|v^2\,dx\\ &\le \varepsilon\|v\|^2 + C_1(\varepsilon)\|u\|_{L^\infty}^2\|v\|^2\\ &\le \varepsilon\int_0^L \frac{(r^m v)_x^2}{\eta}dx + C_1(\varepsilon)\|u\|_{L^\infty}^2\|v\|^2,\end{aligned}$$

which, together with (4.3.43), applying the Gronwall inequality and taking $\varepsilon > 0$ small enough, yields

$$\int_0^t \Big(\|v\|_{H^1}^2 + \|v\|_{L^\infty}^2\Big)(s)\,ds \le C_1. \tag{4.3.56}$$

Similarly, multiplying (4.2.6) by v_t, integrating the resulting equality with respect to x over $[0, L]$ and by virtue of (4.2.6), (4.2.9) and (4.2.11), we know, for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int_0^L \frac{(r^m v)_x^2}{\eta}dx + \int_0^L v_t^2\,dx\\ &\quad = -\int_0^L \frac{\mu}{2}\frac{(r^m v)_x^2}{\eta^2}(r^m u)_x\,dx - \int_0^L \mu m(r^{m-1}uv)\Big(\frac{(r^m v)_x}{\eta}\Big)_x dx - \int_0^L \frac{(uvv_t)_x}{r}dx\\ &\quad = \int_0^L \frac{\mu}{2}r^m u\Big(\frac{(r^m v)_x^2}{\eta^2}\Big)_x dx - \int_0^L m(r^{m-1}uv)\Big(\frac{v_t}{r^m} + \frac{uv}{r^{m+1}}\Big)dx - \int_0^L \frac{(uvv_t)_x}{r}dx\\ &\quad \le \varepsilon\int_0^L v_t^2\,dx + C_1(\varepsilon)\|u\|_{L^\infty}^2\|v\|^2 + C_1(\varepsilon)\|u\|_{L^\infty}^2\int_0^L \frac{(r^m v)_x^2}{\eta}dx,\end{aligned}$$

which, together with (4.3.43) and the Gronwall inequality, and taking $\varepsilon > 0$ small enough, leads to

$$\|v(t)\|_{H^1}^2 + \int_0^t \|v_t(s)\|^2 ds \leq C_1. \tag{4.3.57}$$

Thus we complete the proof of (4.3.54) from (4.3.56) and (4.3.57).

Analogously, we can also use the same method to obtain (4.3.55). Here we omit it. □

Lemma 4.3.7. *Under assumptions in theorem 4.2.1, the following estimate holds for any* $t > 0$,

$$\int_0^t \int_0^L (\sigma^2 u^2 + u^2 u_x^2) dxds \leq C_1 + C_1 \int_0^t \int_0^L \theta^2 u^2 dxds. \tag{4.3.58}$$

Proof. Multiplying (4.2.5) by u^3, integrating the resultant with respect to x over $[0, L]$ and making use of (4.3.54), (4.2.9) and (4.2.11), we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_0^L u^4 dx \\
&\quad = -\int_0^L \left((r^m u^3)_x \Big(-\frac{R\theta}{\eta^p} + \beta \frac{(r^m u)_x}{\eta} \Big) - \frac{v^2 u^3}{r} \right) dx \\
&\quad \leq -C_1 \int_0^L \frac{u^2 u_x^2}{\eta} dx + C_1 \int_0^L (\theta |u|^3 + |u|^3 |u_x| + u^4 + u^2 |u_x| \theta + v^2 |u|^3) dx \\
&\quad \leq -C_1 \int_0^L \frac{u^2 u_x^2}{2\eta} dx + C_1 \int_0^L (u^4 + u^2 \theta^2) dx.
\end{aligned} \tag{4.3.59}$$

Integrating (4.3.59) with respect to t over $[0, t]$, we derive from (4.3.2) and (4.3.43) that

$$\begin{aligned}
\int_0^L u^4 dx + \int_0^t \int_0^L u^2 u_x^2 dxds &\leq C_1 \int_0^t \int_0^L (u^4 + u^2\theta^2) dxds + C_1 \\
&\leq C_1 \int_0^t \max_{x\in[0, L]} u^2 ds + C_1 \int_0^t \int_0^L u^2 \theta^2 dxds + C_1 \\
&\leq C_1 \int_0^t \int_0^L u^2 \theta^2 dxds + C_1
\end{aligned}$$

which implies (4.3.58). □

The next lemma is about to get the bound of η_x which is controlled by θ_x.

Lemma 4.3.8. *Under assumptions in theorem 4.2.1, the following estimate holds for any* $t > 0$ *and* $\varepsilon > 0$,

$$\|\eta_x(t)\|^2 + \int_0^t \|\eta_x(s)\|^2 ds \leq C_1(\varepsilon) + \varepsilon \int_0^t \|\theta_x(s)\|^2 ds. \tag{4.3.60}$$

Proof. Noting (4.2.11), we can obtain from (4.2.4) and (4.2.5) that

$$\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)_t=\left(\frac{R\theta}{\eta^p}\right)_x+\frac{mu^2-v^2}{r^{m+1}}. \tag{4.3.61}$$

Multiplying (4.3.61) by $\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}$, integrating the result with respect to x over $[0, L]$ and using (4.3.54) and (4.2.11), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)^2dx\\
&\quad=\int_0^L\left(\frac{R\theta}{\eta^p}\right)_x\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)dx+\int_0^L\frac{mu^2-v^2}{r^{m+1}}\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)dx\\
&\quad=-\int_0^L\frac{Rp\theta}{\beta\eta^p}\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)^2dx+\int_0^L\left(\frac{R\theta_x}{\eta^p}-\frac{Rp\theta u}{\beta r^m\eta^p}\right)\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)dx\\
&\qquad+\int_0^L\frac{mu^2-v^2}{r^{m+1}}\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)dx\\
&\quad\le-\int_0^L\frac{Rp\theta}{\beta\eta^p}\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)^2dx+C_1\int_0^L(|\theta_x\eta_x|+|\theta_x u|\\
&\qquad+\theta|u\eta_x|+\theta u^2+u^2|\eta_x|+|u^3|+|v^3|+v^2|\eta_x|)\,dx.
\end{aligned} \tag{4.3.62}$$

Integrating (4.3.62) with respect to t over $[0, t]$ and applying (4.2.11), (4.3.1)–(4.3.43) and (4.3.54), we get, for any $\varepsilon>0$, $\delta>0$,

$$\begin{aligned}
&\int_0^L\eta_x^2dx+\int_0^t\int_0^L\theta\frac{\eta_x^2}{\eta}dxds\\
&\quad\le\varepsilon\int_0^t\int_0^L\left(\frac{1}{2}\theta_x^2+\theta\frac{\eta_x^2}{\eta}\right)dxds\\
&\qquad+C_1(\varepsilon)\int_0^t\int_0^L\left(\theta u^2+u^2+\frac{\theta_x^2}{\theta}+u^2|\eta_x|+|u^3|+v^2|\eta_x|\right)dxds+C_1\\
&\quad\le\varepsilon\int_0^t\int_0^L\left(\frac{1}{2}\theta_x^2+\theta\frac{\eta_x^2}{\eta}\right)dxds+C_1(\varepsilon)\int_0^t\int_0^L\frac{r^{2m}\theta_x^2}{\theta^2\eta}dxds\\
&\qquad+C_1(\varepsilon)\int_0^t\max_{x\in[0,\,L]}u^2\int_0^L\theta\,dxds\\
&\qquad+\delta\int_0^t\int_0^L\eta_x^2dxds+C_1(\varepsilon,\delta)\int_0^t\max_{x\in[0,\,L]}u^2\int_0^L u^2dxds+C_1(\varepsilon,\delta)\\
&\quad\le\varepsilon\int_0^t\int_0^L\theta\frac{\eta_x^2}{\eta}dxds+\varepsilon\int_0^t\int_0^L\theta_x^2dxds+\delta\int_0^t\int_0^L\eta_x^2dxds+C_1(\varepsilon,\delta),
\end{aligned} \tag{4.3.63}$$

where in the first inequality we have used the fact

$$\left(\beta\frac{\eta_x}{\eta}-\frac{u}{r^m}\right)^2\ge\frac{1}{2}\left(\beta\frac{\eta_x}{\eta}\right)^2-\frac{u^2}{r^{2m}}. \tag{4.3.64}$$

On the other hand, it follows from (4.3.9) and (4.3.63) that

$$\int_0^L \eta_x^2 dx + \int_0^t \int_0^L \theta \frac{\eta_x^2}{\eta} dxds \le \varepsilon \int_0^t \int_0^L \theta_x^2 dxds + \delta \int_0^t \int_0^L \eta_x^2 dxds \\ + C_1(\varepsilon) \int_0^t V(s) \int_0^L \eta_x^2 dxds + C_1(\varepsilon, \delta). \tag{4.3.65}$$

Setting $\delta = \frac{1}{2}$, using the Gronwall inequality and (4.3.2), we can complete the proof of the lemma. □

In the next lemma, we shall give the uniform estimate of the derivative of θ to close lemma 4.3.8.

Lemma 4.3.9. *Under assumptions in theorem 4.2.1, we have, for any $t > 0$,*

$$\|\theta(t)\|^2 + \int_0^t \|\theta_x(s)\|^2 ds \le C_1. \tag{4.3.66}$$

Proof. Define the entropy of the p-th power Newtonian fluid $S(x,t) = c_v \log\theta + Rh(\eta)$, and $\omega_1 = c_v\theta + \frac{1}{2}(u^2+v^2+w^2) - \overline{\theta}S$. Due to the boundedness of η, we know that there exists a constant $C_1 > 0$ which is large enough such that $\omega_1 \ge c_v\theta - C_1$. Let $\omega = \omega_1 + C_1$. From (4.2.4)–(4.2.8), we have

$$\omega_x = c_v\left(1 - \frac{\overline{\theta}}{\theta}\right)\theta_x + uu_x + vv_x + ww_x - \frac{R\overline{\theta}\eta_x}{\eta^p}, \tag{4.3.67}$$

and

$$\omega_t + \frac{\eta Q}{\theta} + \frac{kr^{2m}\theta_x^2}{\eta\theta^2} = \Bigg(r^m u\sigma + \mu\frac{r^m v(r^m v)_x}{\eta} + \mu\frac{r^{2m}ww_x}{\eta} \\ -2\mu m(r^{m-1}u^2 + r^{m-1}v^2) - \left(1 - \frac{\overline{\theta}}{\theta}\right)q\Bigg)_x. \tag{4.3.68}$$

Multiplying (4.3.68) by ω, integrating the result with respect to x over $[0, L]$ and using (4.2.9), (4.2.10), (4.3.67) and lemma 4.3.6, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L \omega^2 dx \le -\int_0^L \Bigg(r^m u\sigma + \mu\frac{r^m v(r^m v)_x}{\eta} + \mu\frac{r^{2m}ww_x}{\eta} \\ -2\mu m(r^{m-1}u^2 + r^{m-1}v^2) - \left(1 - \frac{\overline{\theta}}{\theta}\right)q\Bigg) \\ \times \left(c_v\left(1 - \frac{\overline{\theta}}{\theta}\right)\theta_x + uu_x + vv_x + ww_x - \frac{R\overline{\theta}\eta_x}{\eta^p}\right)dx \\ \le -\int_0^L \frac{c_v}{2}\left(1 - \frac{\overline{\theta}}{\theta}\right)^2 \frac{kr^{2m}\theta_x^2}{\eta} dx \\ + C_1\int_0^L (u^3|u_x| + |\eta_x|u^2 + u^4 + u^2u_x^2 + \eta_x^2 + \sigma^2u^2 + v_x^2 + w_x^2)dx. \tag{4.3.69}$$

By (4.3.2), we derive

$$\begin{aligned}\int_0^t\int_0^L \frac{c_v}{2}\left(1-\frac{\overline{\theta}}{\theta}\right)^2\frac{kr^{2m}\theta_x^2}{\eta}\,dxds &\geq \int_0^t\int_0^L \frac{c_v}{2}\left(\frac{1}{2}-\frac{\overline{\theta}^2}{\theta^2}\right)\frac{kr^{2m}\theta_x^2}{\eta}\,dxds\\ &\geq C_1\int_0^t\int_0^L \theta_x^2\,dxds - C_1^{-1}\int_0^t\int_0^L \frac{kr^{2m}\theta_x^2}{\eta\theta^2}dxds\\ &\geq C_1\int_0^t\int_0^L \theta_x^2\,dxds - C_1^{-1},\end{aligned}$$

which, together with lemmas 4.3.7, 4.3.8, (4.3.69) and $\omega \geq c_v\theta$, implies, for any $\varepsilon > 0$,

$$\begin{aligned}&\int_0^L \theta^2\,dx + \int_0^t\int_0^L \theta_x^2\,dxds\\ &\leq C_1\int_0^t\int_0^L (u^2u_x^2+\eta_x^2+\sigma^2u^2)\,dxds + C_1\\ &\leq C_1\int_0^t\int_0^L (u^2\theta^2+\eta_x^2)\,dxds + C_1\\ &\leq C_1\int_0^t\int_0^L u^2\theta^2\,dxds + \varepsilon\int_0^t\int_0^L \theta_x^2\,dxds + C_1(\varepsilon)\\ &\leq C_1\int_0^t\left(\max_{x\in[0,\,L]} u^2\int_0^L \theta^2\,dx\right)ds + \varepsilon\int_0^t\int_0^L \theta_x^2\,dxds + C_1(\varepsilon). \end{aligned}\tag{4.3.70}$$

Using the Gronwall inequality and (4.3.43), we obtain (4.3.66). □

By lemma 4.3.6, we can get the uniform estimate of the derivative of η.

Lemma 4.3.10. *Under assumptions in theorem 4.2.1, we have, for any* $t > 0$,

$$\|\eta_x(t)\|^2 + \int_0^t\left(\|\eta_x\|^2+\|\theta\eta_x\|^2+\|u_x\|^2\right)(s)\,ds \leq C_1. \tag{4.3.71}$$

Proof. First, it follows from lemmas 4.3.8 and 4.3.9 that

$$\|\eta_x(t)\|^2 + \int_0^t \|\eta_x(s)\|^2\,ds \leq C_1. \tag{4.3.72}$$

Next, we derive from the Sobolev inequality and (4.3.3) that

$$\theta^2 \leq C_1\left(\int_0^L \theta\,dx + \int_0^L |\theta_x|\,dx\right)^2 \leq C_1 + C_1\int_0^L |\theta_x|^2\,dx,$$

which, together with lemma 4.3.9 and (4.3.72), yields

$$\begin{aligned}\int_0^t\int_0^L \theta^2\eta_x^2\,dxds &\le \int_0^t\left(\max_{x\in[0,\,L]}\theta^2\int_0^L\eta_x^2dx\right)ds\\ &\le C_1\int_0^t\int_0^L\eta_x^2dxds + C_1\int_0^t\left(\int_0^L\eta_x^2dx\right)\left(\int_0^L\theta_x^2dx\right)ds\\ &\le C_1. \end{aligned} \tag{4.3.73}$$

Finally, multiplying (4.2.5) by u, integrating the result over $[0,\,L]\times[0,\,t]$ and using lemmas 4.3.6, 4.3.9, (4.2.11), (4.3.43) and (4.3.73), we have

$$\int_0^L u^2\,dx + \int_0^t\int_0^L u_x^2dxds \le C_1\int_0^t\int_0^L(\theta_x^2+u^2+\theta^2\eta_x^2+uv^2)\,dxds\le C_1.$$

Thus we complete the proof of the lemma. □

The following two lemmas are the high-order estimates of η, u, v, w and θ.

Lemma 4.3.11. *Under assumptions in theorem 4.2.1, we have, for any* $t>0$,

$$\|v_x(t)\|^2+\int_0^t\left(\|v_{xx}\|^2+\|\theta v_x\|^2+\|\eta_x v_x\|^2+\|v_x\|_{L^4}^4\right)(s)\,ds\le C_1, \tag{4.3.74}$$

$$\|w_x(t)\|^2+\int_0^t\left(\|w_{xx}\|^2+\|\theta w_x\|^2+\|\eta_x w_x\|^2+\|w_x\|_{L^4}^4\right)(s)\,ds\le C_1, \tag{4.3.75}$$

$$\|u_x(t)\|^2+\int_0^t\left(\|u_{xx}\|^2+\|\theta u_x\|^2+\|\eta_x u_x\|^2+\|u_x\|_{L^4}^4\right)(s)\,ds\le C_1. \tag{4.3.76}$$

Proof. We derive from the integration by parts, (4.2.6) and (4.2.9) that

$$\begin{aligned}\frac{d}{dt}\int_0^L v_x^2dx &= 2\int_0^L v_xv_{xt}\,dx = -2\int_0^L v_tv_{xx}\,dx\\ &= -2\int_0^L\left(\mu r^m\left(\frac{(r^mv)_x}{\eta}\right)_x v_{xx}-\frac{uv}{r}v_{xx}\right)dx\\ &\le -\int_0^L\frac{\mu r^{2m}v_{xx}^2}{\eta}dx + C_1\int_0^L(v_x^2+v^2\eta_x^2+v_x^2\eta_x^2+v^2+u^2v^2)\,dx. \end{aligned}\tag{4.3.77}$$

Integrating (4.3.77) with respect to t over $[0,\,t]$ and using lemmas 4.3.6, 4.3.10 and (4.3.43), we have for any $\varepsilon>0$,

$$\begin{aligned}\int_0^L v_x^2dx+\int_0^t\int_0^L v_{xx}^2dxds &\le C_1\int_0^t\int_0^L\eta_x^2v_x^2dxds + C_1\\ &\le C_1+\varepsilon\int_0^t\left(\int_0^L v_{xx}^2dx\right)\left(\int_0^L\eta_x^2dx\right)ds\\ &\quad + C_1(\varepsilon)\int_0^t\left(\int_0^L v_x^2dx\right)\left(\int_0^L\eta_x^2dx\right)ds\end{aligned}$$

where we have used the fact,

$$\max_{x\in[0,\,L]} v_x^2 \le \varepsilon \int_0^L v_{xx}^2\,dx + C_1(\varepsilon)\int_0^L v_x^2\,dx.$$

Moreover, by lemma 4.3.10, we get

$$\int_0^L v_x^2\,dx + \int_0^t\int_0^L v_{xx}^2\,dxds \le C_1. \tag{4.3.78}$$

Using lemma 4.3.10 and (4.3.78), we have

$$\begin{aligned}\int_0^t\int_0^L (\theta^2 v_x^2 + \eta_x^2 v_x^2 + v_x^4)\,dxds &\le C_1\int_0^t \left(\max_{x\in[0,\,L]} v_x^2 \int_0^L (\theta^2+\eta_x^2+v_x^2)\,dx\right) ds\\ &\le C_1\int_0^t \max_{x\in[0,\,L]} v_x^2\,ds\\ &\le C_1\int_0^t\int_0^L v_x^2\,dxds + C_1\int_0^t\int_0^L v_{xx}^2\,dxds\\ &\le C_1,\end{aligned}$$

which, along with (4.3.78), leads to (4.3.74). Analogously, we also get (4.3.75) from (4.2.7) and lemmas 4.3.6 and 4.3.10.

It follows from the integration by parts, (4.2.5) and (4.2.9) that

$$\begin{aligned}\frac{d}{dt}\int_0^L u_x^2\,dx &= 2\int_0^L u_x u_{xt}\,dx = -2\int_0^L u_t u_{xx}\,dx\\ &= -2\int_0^L \left(r^m u_{xx}\Big(-\frac{R\theta}{\eta^p} + \frac{\beta(r^m u)_x}{\eta}\Big)_x + \frac{v^2}{r}u_{xx}\right)dx\\ &\le -\int_0^L \frac{\beta r^{2m}u_{xx}^2}{\eta}dx + C_1\int_0^L (\theta_x^2 + \theta^2\eta_x^2 + u_x^2 + u^2\eta_x^2 + u_x^2\eta_x^2 + u^2 + v^4)\,dx.\end{aligned} \tag{4.3.79}$$

We thus integrate (4.3.79) with respect to t over $[0,\,t]$ and make use of lemmas 4.3.6, 4.3.9, 4.3.10 and (4.3.43) to obtain for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^L u_x^2\,dx + \int_0^t\int_0^L u_{xx}^2\,dxds &\le C_1\int_0^t\int_0^L \eta_x^2 u_x^2\,dxds + C_1\\ &\le C_1 + \varepsilon\int_0^t \left(\int_0^L u_{xx}^2\,dx\right)\left(\int_0^L \eta_x^2\,dx\right)ds\\ &\quad + C_1(\varepsilon)\int_0^t \left(\int_0^L u_x^2\,dx\right)\left(\int_0^L \eta_x^2\,dx\right)ds\end{aligned}$$

where we have noted that

$$\max_{x\in[0,\,L]} u_x^2 \le \varepsilon\int_0^L u_{xx}^2\,dx + C_1(\varepsilon)\int_0^L u_x^2\,dx.$$

On the other hand, by lemma 4.3.10, we know

$$\int_0^L u_x^2 dx + \int_0^t \int_0^L u_{xx}^2 dxds \leq C_1. \tag{4.3.80}$$

Using lemma 4.3.10 and (4.3.80), we have

$$\begin{aligned}\int_0^t \int_0^L (\theta^2 u_x^2 + \eta_x^2 u_x^2 + u_x^4) dxds &\leq C_1 \int_0^t \left(\max_{x\in[0,\, L]} u_x^2 \int_0^L (\theta^2 + \eta_x^2 + u_x^2) dx \right) ds \\ &\leq C_1 \int_0^t \max_{x\in[0,\, L]} u_x^2 ds \\ &\leq C_1 \int_0^t \int_0^L u_x^2 dxds + C_1 \int_0^t \int_0^L u_{xx}^2 dxds \\ &\leq C_1\end{aligned}$$

which, along with (4.3.80), leads to (4.3.76). Thus we complete the proof of the lemma. □

Lemma 4.3.12. *Under assumptions in theorem 4.2.1, the following estimates hold for any* $t > 0$,

$$\|\theta_x(t)\|^2 + \int_0^t \left(\|\theta_{xx}\|^2 + \|\eta_x \theta_x\|^2 \right)(s) ds \leq C_1, \tag{4.3.81}$$

$$\int_0^t \left(\|\eta_t\|^2 + \|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \|\theta_t\|^2 \right)(s) ds \leq C_1. \tag{4.3.82}$$

Proof. Integrating by parts and using (4.2.8) and (4.2.10), we have

$$\begin{aligned}\frac{d}{dt}\int_0^L \theta_x^2 dx &= 2\int_0^L \theta_x \theta_{xt} dx = -2\int_0^L \theta_t \theta_{xx} dx \\ &= -\frac{2}{c_v}\int_0^L \theta_{xx}\Big(\Big(-\frac{R\theta}{\eta^p} + \frac{\beta(r^m u)_x}{\eta}\Big)(r^m u)_x + \Big(\frac{kr^{2m}\theta_x}{\eta}\Big)_x \\ &\quad + \frac{\mu(r^m v)_x^2}{\eta} + \frac{\mu r^{2m} w_x^2}{\eta} - 2\mu m(r^{m-1}u^2 + r^{m-1}v^2)_x\Big) dx \\ &\leq -\frac{1}{c_v}\int_0^L \frac{kr^{4m}\theta_{xx}^2}{\eta} dx \\ &\quad + C_1 \int_0^L (\theta_x^2 + \theta^2 u_x^2 + \theta^2 \eta_x^2 + u_x^4 + v_x^4 + w_x^4 + u^2 u_x^2 + u^4 + u^2\theta^2) dx.\end{aligned} \tag{4.3.83}$$

We thus integrate (4.3.83) with respect to t over $[0,\, t]$ and derive from lemmas 4.3.6, 4.3.9, 4.3.10 and (4.3.43) to obtain for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^L \theta_x^2 dx + \int_0^t \int_0^L \theta_{xx}^2 dxds &\le C_1 \int_0^t \int_0^L (\theta_x^2 + \theta^2 u_x^2 + \eta_x^2\theta_x^2 + u_x^4 + u^2 u_x^2) dxds + C_1 \\ &\le C_1 + \varepsilon \int_0^t \left(\int_0^L \theta_{xx}^2 dx\right)\left(\int_0^L \eta_x^2 dx\right) ds \\ &\quad + C_1(\varepsilon) \int_0^t \left(\int_0^L \theta_x^2 dx\right)\left(\int_0^L \eta_x^2 dx\right) ds\end{aligned}$$

where we have noted that,

$$\max_{x\in[0,\,L]} \theta_x^2 \le \varepsilon \int_0^L \theta_{xx}^2 dx + C_1(\varepsilon) \int_0^L \theta_x^2 dx.$$

Finally, using lemma 4.3.10 and (4.3.76), we easily obtain (4.3.81). Estimate (4.3.82) can easily be derived from lemmas 4.3.6–4.3.11, (4.3.81) and equations (4.2.4)–(4.2.8). Thus we complete the proof of the lemma. □

The next lemma concerns the uniform lower and upper bounds of absolute temperature $\theta(x,\,t)$.

Lemma 4.3.13. *Under assumptions in theorem 4.2.1, we have for all* $(x,\,t) \in [0,\,L] \times [0,\,+\infty)$,

$$0 < C_1^{-1} \le \theta(x,t) \le C_1. \tag{4.3.84}$$

Proof. By the interpolation inequality, we derive from lemmas 4.3.1 and 4.3.12 that

$$\|\theta(t)\|_{L^\infty} \le C_1 \|\theta_x\|^{\frac{2}{3}} \|\theta\|_{L^1}^{\frac{1}{3}} + C_1 \|\theta\|_{L^1} \le C_1$$

which implies

$$\theta(x,t) \le C_1.$$

Similarly to the proof of (2.3.110) in lemma 2.3.12, we can obtain the uniform lower bound of the absolute temperature θ. Thus we complete the proof. □

4.3.2 Exponential Stability in H^1

The next lemma concerns on exponential stability of solutions to problem (4.2.4)–(4.2.10).

Lemma 4.3.14. *There exists a positive constant* $\gamma_1 = \gamma_1(C_1) > 0$ *such that for any fixed* $\gamma \in (0,\,\gamma_1]$, *the following estimate holds for any* $t > 0$,

$$\begin{aligned}&e^{\gamma t}\Big(\|\eta(t)-\overline{\eta}\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 + \big\|\theta(t)-\overline{\theta}\big\|_{H^1}^2 \\ &\quad + \|r(t)-\overline{r}\|_{H^2}^2 + \|r_t(t)\|_{H^1}^2\Big) + \int_0^t e^{\gamma s}\Big(\|\eta-\overline{\eta}\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|w\|_{H^2}^2 \\ &\quad + \big\|\theta-\overline{\theta}\big\|_{H^2}^2 + \|r-\overline{r}\|_{H^2}^2 + \|r_t\|_{H^2}^2 + \|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \|\theta_t\|^2\Big)(s)ds \le C_1.\end{aligned} \tag{4.3.85}$$

Proof. Similarly to the proofs of lemmas 2.3.13–2.3.15 in chapter 2, we can prove this lemma. See also lemmas 3.1 and 3.2 in Qin [103] or, *e.g.*, [18]. ☐

Till now we have completed the proof of theorem 4.2.1. ☐

4.4 Global Existence and Exponential Stability in H^2

In this section, we shall show theorem 4.2.2, that is, the global existence and exponential stability of solutions in H^2 to the problem (4.2.4)–(4.2.10).

Lemma 4.4.1. *Let* $(\eta_0, u_0, v_0, w_0, \theta_0) \in (H^2[0, L])^5$, *then the problem (4.2.4)–(4.2.10) admits a unique global solution* $(\eta(t), u(t), v(t), w(t), \theta(t))$ *in* H^2. *Moreover, there exists a positive constant* $\gamma_2 = \gamma_2(C_2) \le \gamma_1$ *such that for any* $\gamma \in [0, \gamma_2]$, *the following estimate holds for any* $t \ge 0$,

$$\begin{aligned}&e^{\gamma t}\Big(\|\eta(t)-\overline{\eta}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|v(t)\|_{H^2}^2+\|w(t)\|_{H^2}^2+\big\|\theta(t)-\overline{\theta}\big\|_{H^2}^2\Big)\\&\quad+\int_0^t\Big(e^{\gamma s}\|\eta-\overline{\eta}\|_{H^2}^2+\|u\|_{H^3}^2+\|v\|_{H^3}^2+\|w\|_{H^3}^2+\big\|\theta-\overline{\theta}\big\|_{H^3}^2\\&\quad+\|u_t\|_{H^1}^2+\|v_t\|_{H^1}^2+\|w_t\|_{H^1}^2+\|\theta_t\|_{H^1}^2\Big)(s)\,ds\le C_2.\end{aligned}\tag{4.4.1}$$

Proof. Differentiating (4.2.5) with respect to t and multiplying the resultant by $u_t e^{\gamma t}$, we derive from theorem 4.2.1 that for $\gamma > 0$ small enough,

$$\begin{aligned}&e^{\gamma t}\|u_t(t)\|^2+C_1^{-1}\int_0^t e^{\gamma s}\big\|(r^m u)_{xt}(s)\big\|^2 ds\\&\le C_2+C_2\gamma\int_0^t e^{\gamma s}\big\|(r^m u)_{xt}(s)\big\|^2 ds\\&\quad+C_1\int_0^t e^{\gamma s}\Big(\|u\|_{H^2}^2+\|u_t\|^2+\|\theta_t\|^2+\|v_t\|^2+\|v\|^2\Big)(s)\,ds\\&\le C_2+C_2\gamma\int_0^t e^{\gamma s}\big\|(r^m u)_{xt}(s)\big\|^2 ds\end{aligned}$$

which leads to

$$e^{\gamma t}\|u_t(t)\|^2+\int_0^t e^{\gamma s}\big\|(r^m u)_{xt}(s)\big\|^2 ds\le C_2,\quad \gamma\in[0,\gamma_2']\tag{4.4.2}$$

where $\gamma_2' = \gamma_2'\,(C_2) \le \gamma_1 > 0$ is a constant. By (4.2.5) and theorem 4.2.1, we know

$$\|u_{xx}(t)\|\le C_1\Big(\|u_t\|+\big\|\theta-\overline{\theta}\big\|_{H^1}+\|\eta-\overline{\eta}\|_{H^1}+\|u\|_{H^1}+\|v\|_{H^1}+\|w\|_{H^1}+\|\theta\|_{H^1}\Big)$$

which, together with (4.4.2) and theorem 4.2.1, yields

$$e^{\gamma t}\Big(\|u_t(t)\|^2+\|u(t)\|_{H^2}^2\Big)+\int_0^t e^{\gamma s}\Big(\|u_t\|_{H^1}^2+\|u\|_{H^3}^2\Big)(s)\,ds\le C_2. \tag{4.4.3}$$

Similarly, we easily get

$$\begin{aligned}&e^{\gamma t}\Big(\|v_t(t)\|^2+\|w_t(t)\|^2+\|v(t)\|_{H^2}^2+\|w(t)\|_{H^2}^2\Big)\\&\quad+\int_0^t e^{\gamma s}\Big(\|v_t\|_{H^1}^2+\|w_t\|_{H^1}^2+\|v\|_{H^3}^2+\|w\|_{H^3}^2\Big)(s)\,ds\le C_2.\end{aligned} \tag{4.4.4}$$

Differentiating (4.2.5) with respect to x, using (4.2.4), and multiplying the result by $\frac{\eta_{xx}}{\eta}$ in $L^2[0,\,L]$, we derive that

$$\begin{aligned}\frac{d}{dt}\left\|\frac{\eta_{xx}}{\eta}\right\|^2+C_1^{-1}\left\|\frac{\eta_{xx}}{\eta}\right\|^2\le\frac{1}{2C_1}\left\|\frac{\eta_{xx}}{\eta}\right\|^2+C_1\Big(\|u_{xt}\|^2+\|\theta_x\|^2\\+\big\|(r^m u)_{xx}\big\|^2+\|\eta_x\|^2+\|\theta_{xx}\|^2+\|v_x\|^2\Big).\end{aligned} \tag{4.4.5}$$

Multiplying (4.4.5) by $e^{\gamma t}$, using theorem 4.2.1 and (4.4.2)–(4.4.4), there exists a constant $\gamma_2''=\gamma_2''\,(C_2)\le\gamma_2'\le\gamma_1$ such that when $\gamma\in[0,\,\gamma_2'']$,

$$e^{\gamma t}\|\theta_{xx}(t)\|^2+\int_0^t e^{\gamma s}\|\theta_{xx}(s)\|^2\,ds\le C_2. \tag{4.4.6}$$

Differentiating (4.2.8) with respect to t and multiplying it by $\theta_t e^{\gamma t}$, we have

$$\begin{aligned}e^{\gamma t}\|\theta_t(t)\|^2+\int_0^t e^{\gamma s}\|\theta_{xt}(s)\|^2\,ds&\le C_2+C_2\gamma\int_0^t e^{\gamma s}\|\theta_{xt}(s)\|^2\,ds+C_1\int_0^t e^{\gamma s}\Big(\|u\|_{H^2}^2+\|\theta_t\|^2\\&\quad+\|\eta_t\|^2+\|u_t\|_{H^1}^2+\|v_t\|_{H^1}^2+\|w_t\|_{H^1}^2+\|\theta_x\|^2\Big)(s)\,ds\\&\le C_2+C_2\gamma\int_0^t e^{\gamma s}\|\theta_{xt}(s)\|^2\,ds,\end{aligned}$$

that is,

$$e^{\gamma t}\|\theta_t(t)\|^2+\int_0^t e^{\gamma s}\|\theta_{xt}(s)\|^2\,ds\le C_2,\quad\forall\gamma\in[0,\gamma_2] \tag{4.4.7}$$

where $0<\gamma_2=\gamma_2(C_2)\le\gamma_2''\le\gamma_1$ is a constant. By (4.2.8), we have

$$\|\theta_{xx}(t)\|\le C_1\Big(\|\theta_t\|+\|\eta-\overline{\eta}\|_{H^1}+\|u\|_{H^1}+\|v\|_{H^1}+\|w\|_{H^1}+\big\|\theta-\overline{\theta}\big\|_{H^1}\Big).$$

Thus we have

$$e^{\gamma t}\Big(\|\theta_t(t)\|^2+\|\theta(t)\|_{H^2}^2\Big)+\int_0^t e^{\gamma s}\Big(\|\theta_t\|_{H^1}^2+\|\theta\|_{H^3}^2\Big)(s)\,ds\le C_2 \tag{4.4.8}$$

which, together with (4.4.3) and (4.4.4), leads to (4.4.1). □

Till now we have completed the proof of theorem 4.2.2. □

4.5 Global Existence and Exponential Stability in H^4

In this section, we shall prove theorem 4.2.3, that is, the global existence and exponential stability of solutions in H^4 to the problem (4.2.4)–(4.2.10).

Lemma 4.5.1. *Under assumptions of theorem 4.2.3, there holds that for any* $t > 0$ *and* $\varepsilon > 0$ *small enough,*

$$\|u_{xt}(x,0)\| + \|v_{xt}(x,0)\| + \|w_{xt}(x,0)\| + \|\theta_{xt}(x,0)\| \le C_4, \tag{4.5.1}$$

$$\|u_{tt}(x,0)\| + \|v_{tt}(x,0)\| + \|w_{tt}(x,0)\| + \|\theta_{tt}(x,0)\| \le C_4, \tag{4.5.2}$$

$$\|u_{txx}(x,0)\| + \|v_{txx}(x,0)\| + \|w_{txx}(x,0)\| + \|\theta_{txx}(x,0)\| \le C_4, \tag{4.5.3}$$

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|\theta_{txx}\|^2 + \|u_{txx}\|^2 + \|v_{txx}\|^2\right)(s)\, ds, \tag{4.5.4}$$

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|u_{txx}\|^2 + \|v_{txx}\|^2\right)(s)\, ds, \tag{4.5.5}$$

$$\|w_{tt}(t)\|^2 + \int_0^t \|w_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|u_{txx}\|^2 + \|w_{txx}\|^2\right)(s)\, ds, \tag{4.5.6}$$

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{ttx}(s)\|^2 ds \le{}& C_4\varepsilon^{-3} + C_2\varepsilon^{-1} \int_0^t \|\theta_{txx}(s)\|^2 ds \\ &+ C_1\varepsilon \int_0^t \left(\|u_{ttx}\|^2 + \|v_{ttx}\|^2 + \|w_{ttx}\|^2 + \|u_{txx}\|^2\right)(s)\, ds.\end{aligned} \tag{4.5.7}$$

Proof. Similarly to lemma 2.5.1 in chapter 2, we can obtain estimates (4.5.1)–(4.5.7). See also lemma 5.4.1 in Qin [99] or lemma 3.2 in Qin and Jiang [114]. □

Lemma 4.5.2. *Under the assumptions of theorem 4.2.3, the following estimates hold that for any* $t > 0$ *and* $\varepsilon > 0$ *small enough,*

$$\|u_{xt}(t)\|^2 + \int_0^t \|u_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2 \int_0^t \left(\|u_{xtt}\|^2 + \|\theta_{xxt}\|^2\right)(s)\, ds, \tag{4.5.8}$$

$$\|v_{xt}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2 \int_0^t \|v_{xtt}(s)\|^2 ds, \tag{4.5.9}$$

$$\|w_{xt}(t)\|^2 + \int_0^t \|w_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2 \int_0^t \|w_{xtt}(s)\|^2 ds, \tag{4.5.10}$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 ds \le C_4 + C_2\varepsilon^2 \int_0^t \Big(\|u_{xxt}\|^2 + \|v_{xxt}\|^2 + \|w_{xxt}\|^2\Big)(s)\, ds. \tag{4.5.11}$$

Proof. Similarly to lemma 2.5.2 in chapter 2, we can easily prove this lemma. See also lemma 5.4.2 in Qin [99] or lemma 2.4 in Qin and Jiang [114]. Here we omit the detail. □

Lemma 4.5.3. *Under the assumptions of theorem 4.2.3, we have for any $t > 0$,*

$$\begin{aligned}&\|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \|w_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|u_{xt}(t)\|^2 + \|v_{xt}(t)\|^2\\ &\quad + \|w_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t \Big(\|u_{ttx}\|^2 + \|v_{ttx}\|^2 + \|w_{ttx}\|^2 + \|\theta_{ttx}\|^2\\ &\quad + \|u_{xxt}\|^2 + \|v_{xxt}\|^2 + \|w_{xxt}\|^2 + \|\theta_{xxt}\|^2\Big)(s)\, ds \le C_4,\end{aligned} \tag{4.5.12}$$

$$\begin{aligned}&\|\eta_{xxx}(t)\|_{H^1}^2 + \|u_{xxx}(t)\|_{H^1}^2 + \|v_{xxx}(t)\|_{H^1}^2 + \|w_{xxx}(t)\|_{H^1}^2 + \|\theta_{xxx}(t)\|_{H^1}^2 + \|u_{txx}(t)\|^2\\ &\quad + \|v_{txx}(t)\|^2 + \|w_{txx}(t)\|^2 + \|\theta_{txx}(t)\|^2 + \int_0^t \Big(\|u_{tt}\|^2 + \|v_{tt}\|^2 + \|w_{tt}\|^2 + \|\theta_{tt}\|^2\\ &\quad + \|u_{xxt}\|_{H^1}^2 + \|v_{xxt}\|_{H^1}^2 + \|w_{xxt}\|_{H^1}^2 + \|\theta_{xxt}\|_{H^1}^2\Big)(s)\, ds \le C_4,\end{aligned} \tag{4.5.13}$$

$$\int_0^t \Big(\|\eta_{xxx}\|_{H^1}^2 + \|u_{xxxx}\|_{H^1}^2 + \|v_{xxxx}\|_{H^1}^2 + \|w_{xxxx}\|_{H^1}^2 + \|\theta_{xxxx}\|_{H^1}^2\Big)(s)\, ds \le C_4. \tag{4.5.14}$$

Proof. Similarly to lemma 2.5.3 in chapter 2 or lemmas 5.4.3–5.4.5 in Qin [99] or lemmas 2.5–2.7 in Qin and Jiang [114]. □

Based on lemmas 4.5.1–4.5.3, we can easily establish the exponential stability of global solutions in H^4 to problem (4.2.4)–(4.2.10), similarly to lemmas 2.5.4–2.5.6 in chapter 2 or lemmas 5.4.6–5.4.11 in Qin [99] or lemmas 3.1–3.6 in Qin and Jiang [114]. Here we omit the details.

Till now we have completed the proof of theorem 4.2.3. □

4.6 Bibliographic Comments

In the one-dimensional case, the existence and exponential stability of a unique global classical solution for full compressible Navier–Stokes equations with large initial data are known (see [57, 66, 72] and references therein). In the multi-dimensional case, the global existence of weak solutions with large initial data has been made (see [39, 81] and references therein). In general domain, the global classical solutions of the full Navier–Stokes equations have also been investigated provided that the initial data are small, see [52, 54, 87] and references therein. Moreover, Jiang [65] obtained the large-time behavior of spherically symmetric solutions in multi-dimension. Zheng and Qin [99, 144] established the exponential stability of solutions in $H^i(i = 1, 2, 4)$ and the existence of universal attractors in bounded annular domains. Qin [99, 103] got the large-time behavior of cylindrically symmetric solutions when the initial total energy was sufficiently small $(m = 1)$.

However, for the p-th power Newtonian fluid, there are few results on global solvability and stability. For the large-time behavior in one-dimension space, Lewicka and Watson [79] considered initial-boundary value problems involving fixed endpoints held at a prescribed temperature or insulated and proved exponential convergence of solutions to equilibria for generic initial data. The estimates for different boundary conditions are presented in a unified manner by utilising the thermodynamic concept of availability. Qin and Huang [111] studied the global existence and exponential stability of solutions in $H^i(i = 2, 4)$ for the compressible Navier–Stokes equations, which arise in the study of a thermal explosion and describe the dynamic combustion for a reactive Newtonian fluid. Cui and Yao [18] studied the large-time behavior of the global spherically and cylindrically symmetric solutions in H^1 for the p-th power compressible Newtonian fluid in multi-dimension with large initial data, but there is still a defect in estimating the uniform upper bound of the specific volume. Wang [138] studied the initial and initial-boundary value problems for the p-th power Newtonian fluid with general large initial data and established the existence and uniqueness of globally smooth non-vacuum solutions when the thermal conductivity is some non-negative power of the temperature based on some detailed estimates on the bounds of both density and temperature.

Chapter 5

Global Existence and Asymptotic Behavior of Spherically Symmetric Solutions for the Multi-Dimensional Infrarelativistic Model

5.1 Introduction

In this chapter, we shall consider the motion of the compressible multi-dimensional viscous gas with radiation. The importance of thermal radiation in physical problems increases as the temperature is raised. The role of the radiation is primarily one of transporting energy by radiative process at the moderate temperature, while the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities at the higher temperature. So the radiation field significantly affects the dynamics of the field. The theory of radiation hydrodynamics finds a wide range of applications, such as stellar atmospheres and envelopes, supernova explosions, stellar winds, physics of laser fusion, reentry of vehicles and many others. As will be seen below, the general equations of radiation hydrodynamics are a system of the Euler equations (hyperbolic) coupled with a transport equation (Boltzmann equation). Therefore, the study of mathematical theory of radiation hydrodynamics is of great importance from both the mathematical theory and that of applications. In this chapter, we shall consider a system of the Navier–Stokes equations coupled with a transport equation.

We know the energy in the radiation field to be carried by point, massless particles called photons, which are travelling at speed c of light, characterized by their frequency ϵ, and their energy of each photon $E = h\epsilon$ (where h is the Planck's constant), the momentum $\overrightarrow{p} = \frac{h\epsilon}{c}\overrightarrow{\Omega}$, where $\overrightarrow{\Omega}$ is a unit vector and denotes the direction of travel of the photon (it requires two angular variables to specify $\overrightarrow{\Omega}$). In a radiative transfer, it is conventional to introduce the specific radiative intensity $I \equiv I(x, t, \epsilon, \overrightarrow{\Omega})$ driven by the so-called radiative transfer integro-differential equation introduced and discussed by Chandrasekhar [10].

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Meanwhile, we can derive global quantities by integrating with respect to the angular and frequency variables: the specific radiative energy density $E_R(x, t)$ per unit volume is then $E_R(x,t) = \frac{1}{c}\int\int I(x,t,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon$, and the specific radiative flux $\overrightarrow{F}_R(x,t) = \int\int \overrightarrow{\mathbf{\Omega}} I(x,t,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon$. Under consideration of the three basic interactions between photons and matter, namely, absorption, scattering and emission, we find the transfer in the conventional form (see, *e.g.*, [88, 96, 97])

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\epsilon,\overrightarrow{\mathbf{\Omega}})}{\partial t} + \overrightarrow{\mathbf{\Omega}}\cdot\nabla I(\epsilon,\overrightarrow{\mathbf{\Omega}}) = S_e(\epsilon) - \sigma_a(\epsilon) I(\epsilon,\overrightarrow{\mathbf{\Omega}})& \\ + \int_0^{+\infty} d\epsilon' \int_{S^{n-1}} \Big(\frac{\epsilon}{\epsilon'}\sigma_s(\epsilon'\to\epsilon, \overrightarrow{\mathbf{\Omega}}'\cdot\overrightarrow{\mathbf{\Omega}}) I(\epsilon',\overrightarrow{\mathbf{\Omega}}')& \\ - \sigma_s(\epsilon\to\epsilon', \overrightarrow{\mathbf{\Omega}}\cdot\overrightarrow{\mathbf{\Omega}}') I(\epsilon,\overrightarrow{\mathbf{\Omega}})\Big)\,d\overrightarrow{\mathbf{\Omega}}',& \end{aligned} \tag{5.1.1}$$

where $I(\epsilon,\overrightarrow{\mathbf{\Omega}}) \equiv I(x,t,\overrightarrow{\mathbf{\Omega}},\epsilon)$, S^{n-1} is the unit ball in $\mathbb{R}^n$, $S_e(\epsilon) \equiv S_e(x, t, \epsilon, \rho, \theta)$, $\sigma_a(\epsilon) \equiv \sigma_a(x, t, \epsilon, \rho, \theta)$ and $\sigma_s(\epsilon) \equiv \sigma_s(x, t, \epsilon, \rho, \theta)$, respectively, denote the rate of energy emission due to spontaneous processes, the absorption coefficient and the scattering coefficient that also depend on mass density ρ and temperature θ of the matter. The scattering interaction serves to change the photon's characteristics ϵ' and $\overrightarrow{\mathbf{\Omega}}'$ to a new set of characteristics ϵ and $\overrightarrow{\mathbf{\Omega}}$. The sign $\epsilon' \to \epsilon$ stands for from ϵ' to ϵ and $\overrightarrow{\mathbf{\Omega}}'\cdot\overrightarrow{\mathbf{\Omega}}$ denotes the transfer from direction $\overrightarrow{\mathbf{\Omega}}'$ to direction $\overrightarrow{\mathbf{\Omega}}$ as an argument of $\sigma_s(\epsilon)$. Therefore, we can describe the scattering event by a probabilistic statement concerning this change as follows

$$\text{outscattering} = \int_0^{+\infty} d\epsilon' \int_{S^{n-1}} \sigma_s(\epsilon\to\epsilon', \overrightarrow{\mathbf{\Omega}}\cdot\overrightarrow{\mathbf{\Omega}}') I(\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}',$$

$$\text{inscattering} = \int_0^{+\infty} d\epsilon' \int_{S^{n-1}} \sigma_s(\epsilon'\to\epsilon, \overrightarrow{\mathbf{\Omega}}'\cdot\overrightarrow{\mathbf{\Omega}}) I(\epsilon',\overrightarrow{\mathbf{\Omega}}')\,d\overrightarrow{\mathbf{\Omega}}'.$$

Suppose that the matter is in local thermodynamical equilibrium and radiation is present with coupling terms between matter and radiation, the coupled system reads (see, e.g., [96, 97])

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\mathbf{U}) = 0, & (5.1.2)\\ \partial_t(\rho\mathbf{U}) + \mathrm{div}(\rho\mathbf{U}\otimes\mathbf{U}) = -\mathrm{div}\overrightarrow{\mathbf{\Pi}} - \overrightarrow{S}_F, & (5.1.3)\\ \partial_t(\rho e + \mathrm{div}(\rho e\mathbf{U}) = -\mathrm{div}Q - \overrightarrow{D} : \overrightarrow{\mathbf{\Pi}} - S_E, & (5.1.4)\\ \dfrac{1}{c}\dfrac{\partial I(\epsilon,\overrightarrow{\mathbf{\Omega}})}{\partial t} + \overrightarrow{\mathbf{\Omega}}\cdot\nabla I(\epsilon,\overrightarrow{\mathbf{\Omega}}) = S_t(\epsilon,\overrightarrow{\mathbf{\Omega}}), & (5.1.5)\end{cases}$$

where $\rho = \rho(x,t), \mathbf{U} = \mathbf{U}(x,t), \theta = \theta(x,t), e = e(x,t), Q = Q(x,t)$ are the density, the velocity, the absolute temperature, the internal energy and the heat flux, respectively, $\overrightarrow{\Pi} = -P(\rho,\theta)\overrightarrow{I} + \overrightarrow{\pi}$ represents the material stress tensor for a Newtonian fluid with the viscous contribution $\overrightarrow{\pi} = 2\mu\overrightarrow{D} + \lambda \mathrm{div}\mathbf{U}\overrightarrow{I}$ with $\mu > 0$ and $n\lambda + 2\mu \geq 0$, and the strain tensor $\overrightarrow{D}$ such that $\overrightarrow{D}_{ij} = \frac{1}{2}(\frac{\partial \mathbf{U}_i}{\partial x_j} + \frac{\partial \mathbf{U}_j}{\partial x_i})$, and the coupling terms are

$$\begin{aligned} S_t(\epsilon, \overrightarrow{\Omega}) \equiv S_t(x,t,\epsilon,\overrightarrow{\Omega}) &= \sigma_a(\epsilon,\overrightarrow{\Omega})(B(\epsilon,\theta) - I(\epsilon,\overrightarrow{\Omega})) \\ &\quad + \int_0^{+\infty} d\epsilon' \int_{S^{n-1}} \sigma_s(\epsilon' \to \epsilon, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega})\Big(\frac{\epsilon}{\epsilon'} I(\epsilon', \overrightarrow{\Omega}') \\ &\quad - \sigma_s(\epsilon \to \epsilon', \overrightarrow{\Omega} \cdot \overrightarrow{\Omega}')I(\epsilon,\overrightarrow{\Omega})I(\epsilon',\overrightarrow{\Omega}')\Big)d\overrightarrow{\Omega}', \end{aligned}$$

the radiative energy source

$$S_E(x,t) := \int_0^{+\infty} d\epsilon \int_{S^{n-1}} S_t(\epsilon,\overrightarrow{\Omega})d\overrightarrow{\Omega},$$

the radiative flux

$$\overrightarrow{S}_F(x,t) := \frac{1}{c}\int_0^{+\infty} d\epsilon \int_{S^{n-1}} \overrightarrow{\Omega} S_t(\epsilon,\overrightarrow{\Omega})d\overrightarrow{\Omega},$$

and Planck's function $B(\epsilon, \theta)$ describes the frequency-temperature black body distribution. The thermo-radiative flux Q satisfies Fourier's law

$$Q = Q(\rho,\theta) = -\kappa(\rho,\theta)\nabla\theta \tag{5.1.6}$$

where $\kappa(\rho, \theta)$ is the heat conductivity coefficient.

5.2 Reformulation and Main Theorems

5.2.1 Reformulation of Model

Now we assume $\mathbf{\Omega} \subset \mathbb{R}^n$ is a bounded domain and consider the following boundary conditions

$$\mathbf{U}|_{\partial\Omega} = 0, \quad Q|_{\partial\Omega} = 0, \quad I|_{\partial\Omega} = 0, \tag{5.2.1}$$

together with the initial conditions

$$(\rho, \mathbf{U}, \theta, I)(x,0) = (\rho_0, \mathbf{U}_0, \theta_0, I_0)(x). \tag{5.2.2}$$

We now construct the corresponding system for radial solutions in the Eulerian coordinates. Let $r = |x|$ and take

$$\rho(x,t) = \rho(r,t), \quad \rho\mathbf{U}(x,t) = \rho u(r,t)\frac{x}{r}, \quad \theta(x,t) = \theta(r,t), \quad I(x,t) = I(r,t). \tag{5.2.3}$$

Then we can obtain from the system (5.1.2)–(5.1.5) by the direct calculations

$$\begin{cases} \rho_t+(\rho u)_r+\dfrac{n-1}{r}\rho u=0, & (5.2.4)\\ (\rho u)_t+(\rho u^2)_r+\dfrac{n-1}{r}\rho u^2+P_r=(2\mu+\lambda)\Big(u_r+\dfrac{n-1}{r}u\Big)_r-(S_F)_R, & (5.2.5)\\ (\rho e)_t+(\rho e u)_r+\dfrac{n-1}{r}\rho u e-\kappa_r\theta_r-\kappa\Big(\theta_{rr}+\dfrac{n-1}{r}\theta_r\Big) & \\ \qquad =2\mu\Big(u_r^2+\dfrac{n-1}{r^2}u^2\Big)+\lambda\Big(u_r+\dfrac{n-1}{r}u\Big)^2-P\Big(u_r+\dfrac{n-1}{r}u\Big)-(S_E)_R, & (5.2.6)\\ \dfrac{1}{c}I_t+\omega I_r=S, & (5.2.7) \end{cases}$$

where the domain $\mathbf{\Omega}$ is given by $\mathbf{\Omega}=\{x\in\mathbb{R}^n:\ a<|x|<b\}$ for some constants a and b with $0<a<b<+\infty$, ω denotes the cosine of the angular between position x and direction of travelling $\overrightarrow{\mathbf{\Omega}}$,

$$(S_E)_R(r,t)=\int_0^{+\infty}d\epsilon\int_{S^{n-1}}S(\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$
$$(S_F)_R(r,t)=\frac{1}{c}\int_0^{+\infty}d\epsilon\int_{S^{n-1}}\omega S(\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$

and the initial boundary value conditions (5.2.1) and (5.2.2) read as

$$u|_{r=a,b}=0,\quad \theta_r|_{r=a,b}=0,\quad I|_{r=a,b}=0,\quad t\geq 0, \tag{5.2.8}$$

$$\rho(r,0)=\rho_0(r),\quad u(r,0)=u_0(r),\quad \theta(r,0)=\theta_0(r),\quad I(r,0)=I_0(r),\quad r\in(a,b). \tag{5.2.9}$$

Similarly to chapter 2, we use $v:=\frac{1}{\rho}$ to represent the specific volume and transform the system (5.2.4)–(5.2.7) into the equations in the Lagrangian coordinates as follows,

$$\begin{cases} v_t=(r^{n-1}u)_x,\quad 0<x<L,\ t>0, & (5.2.10)\\ u_t=r^{n-1}\Big(\delta\dfrac{(r^{n-1}u)_x}{v}-P\Big)_x-v(S_F)_R, & (5.2.11)\\ e_t=\Big(\dfrac{r^{2n-2}\kappa\theta_x}{v}\Big)_x+\Big(\delta\dfrac{(r^{n-1}u)_x}{v}-P\Big)(r^{n-1}u)_x-2\mu(n-1)(r^{n-2}u^2)_x-v(S_E)_R, & (5.2.12)\\ \dfrac{1}{c}I_t+\Big(\omega-\dfrac{u}{c}\Big)\dfrac{r^{n-1}}{v}I_x=S, & (5.2.13) \end{cases}$$

with $\delta=\lambda+2\mu$ and the initial boundary conditions (5.2.8) and (5.2.9) become

$$u(x,t)|_{x=0,L}=0,\quad \theta_x(x,t)|_{x=0,L}=0,\quad t\geq 0, \tag{5.2.14}$$

$$I(0,t)=0 \ \text{ for } \ \omega\in(0,1), \quad I(L,t)=0 \ \text{ for } \ \omega\in(-1,0), \quad t\geq 0, \tag{5.2.15}$$

$$v(x,0)=v_0(x), \ \ u(x,0)=u_0(x), \ \ \theta(x,0)=\theta_0(x), \ \ I(x,0)=I_0(x), \ \ x\in\mathbf{\Omega}_n. \tag{5.2.16}$$

Then as we know that $r(x,\,t)$ is determined by

$$r(x,t)=r_0(x)+\int_0^t u(x,s)\,ds, \quad r_0(x):=\left(a^n+n\int_0^x v_0(y)\,dy\right)^{\frac{1}{n}}, \ \ n=2,3,$$

that is,

$$r_t=u, \quad r^{n-1}r_x=v, \quad r|_{x=0}=a, \quad r|_{x=L}=b. \tag{5.2.17}$$

Thus, we can derive from (5.2.17) by the same arguments as those in [99, 109] that for all $(x,\,t)\in[0,\,L]\times[0,\,+\infty)$,

$$0<a=r(0,t)\leq r(x,t)\leq r(L,t)=b. \tag{5.2.18}$$

In this symmetric model, the radiative energy source

$$(S_E)_R\equiv(S_E)_R(x,t)=\int_0^{+\infty}d\epsilon\int_{S^{n-1}}S(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$

the radiative force

$$(S_F)_R\equiv S_F(x,t)=\frac{1}{c}\int_0^{+\infty}d\epsilon\int_{S^{n-1}}\omega S(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$

the source term S is defined as

$$S\equiv S(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})=S_{a,e}(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})+S_s(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}}),$$

where the absorption–emission term is

$$S_{a,e}(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})=\sigma_a(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})(B(x,t;\epsilon)-I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}}))$$

and

$$S_s(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})=\sigma_s(x,t;\epsilon)(\widetilde{I}(x,t;\epsilon)-I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}}))$$

where $\widetilde{I}(x,t;\epsilon)=\frac{1}{\alpha(n)}\int_{S^{n-1}}I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}$, $\alpha(n)$ is the area of unit sphere surface in $\mathbb{R}^n$ and $B(\theta;\epsilon)=B(x,\,t;\,\epsilon)$ is a function of temperature and frequency describing the equilibrium state.

We introduce the radiative energy

$$E_R=\int_0^{+\infty}d\epsilon\int_{S^{n-1}}I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}$$

and the radiative flux

$$F_R = \int_0^{+\infty} d\epsilon \int_{S^{n-1}} \omega I(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) d\overrightarrow{\mathbf{\Omega}}.$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_v(v,\theta) = -P(v,\theta) + \theta P_\theta(v,\theta). \tag{5.2.19}$$

5.2.2 Main Theorems

In this subsection, we give an infrarelativistic model and then state our main results.

If we assume that the fluid motion is small enough with respect to the velocity of light c so that we can drop all the $\frac{1}{c}$ factors in the previous formulation and then get an infrarelativistic model of a compressible Navier–Stokes system coupled to the radiative transfer equation, that is, the system (5.2.10)–(5.2.13) can be rewritten as

$$\begin{cases} v_t = (r^{n-1}u)_x, & (5.2.20)\\ u_t = r^{n-1}\left(\delta \dfrac{(r^{n-1}u)_x}{v} - P\right)_x, & (5.2.21)\\ e_t = \left(\dfrac{r^{2n-2}\kappa\theta_x}{v}\right)_x + \left(\delta\dfrac{(r^{n-1}u)_x}{v} - P\right)(r^{n-1}u)_x - 2\mu(n-1)(r^{n-2}u^2)_x - v(S_E)_R, & (5.2.22)\\ r^{n-1}\omega I_x = vS. & (5.2.23) \end{cases}$$

We assume that e, P, σ_a, σ_s, κ and B are C^{i+1} $(i = 1,\ 2,\ 4)$ functions on $0 < v < +\infty$ and $0 \le \theta < +\infty$ and satisfy the following growth conditions:

(A_1) $e(v,0) \ge 0, \quad c_1^{-1}(1+\theta^s) \le e_\theta(v,\theta) \le c_1(1+\theta^s),$

(A_2) $-c_2^{-1}v^{-2}(1+\theta^{1+s}) \le P_v(v,\theta) \le -c_2 v^{-2}(1+\theta^{1+s}),$

(A_3) $|P_\theta(v,\theta)| \le c_3 v^{-1}(1+\theta^s),$

(A_4) $c_4^{-1}(1+\theta^{1+s}) \le vP(v,\theta) \le c_4(1+\theta^{1+s}), \quad P_v(v,\theta_0) \le 0,$

(A_5) $0 \le P(v,\theta) \le c_5(1+\theta^{1+s}),$

(A_6) $c_6^{-1}(1+\theta^q) \le \kappa(v,\theta) \le c_6(1+\theta^q),$

(A_7) $|\kappa_v(v,\theta)| + |\kappa_{vv}(v,\theta)| \le c_7(1+\theta^q),$

(A_8) $v\sigma_a(v,\theta;\epsilon,\overrightarrow{\mathbf{\Omega}})B^m(\theta;\epsilon) \le c_8|\omega|\theta^{1+\alpha}f(\epsilon,\overrightarrow{\mathbf{\Omega}}), \quad \text{for } m = 1, 2,$

(A_9) $0 < \sigma_a(v,\theta;\epsilon,\overrightarrow{\mathbf{\Omega}}) \le c_9|\omega|^2 g(\epsilon,\overrightarrow{\mathbf{\Omega}}),$

(A_{10}) $(\sigma_a + |(\sigma_a)_v| + |(\sigma_a)_\theta|)(v,\theta;\epsilon,\overrightarrow{\mathbf{\Omega}})(1 + B(\theta;\epsilon) + |B_\theta(\theta;\epsilon)| + |B_{\theta\theta}(\theta;\epsilon)|)$
$\le c_{10}|\omega|h(\epsilon,\overrightarrow{\mathbf{\Omega}}),$

(A_{11}) $0 < \sigma_s(v,\theta;\epsilon) \le c_{11}|\omega|^2 k(\epsilon,\overrightarrow{\mathbf{\Omega}}),$

(A_{12}) $(|(\sigma_a)_{vv}| + |(\sigma_a)_{v\theta}| + |(\sigma_a)_{\theta\theta} + |(\sigma_a)_{vvv}| + |(\sigma_a)_{vv\theta}| + |(\sigma_a)_{v\theta\theta}| + |(\sigma_a)_{\theta\theta\theta}| + |(\sigma_a)_{\theta\theta\theta\theta}|$
$+ |(\sigma_a)_{v\theta\theta\theta}| + |(\sigma_a)_{vv\theta\theta}| + |(\sigma_a)_{vvv\theta}| + |(\sigma_a)_{vvvv}|)(v, \theta; \epsilon, \overrightarrow{\mathbf{\Omega}})(1 + B(\theta; \epsilon) + |B_\theta(\theta; \epsilon)|$
$+ |B_{\theta\theta}(\theta; \epsilon)| + |B_{\theta\theta\theta}(\theta; \epsilon)| + |B_{\theta\theta\theta\theta}(\theta; \epsilon)|) \leq c_{12}|\omega| l(\epsilon, \overrightarrow{\mathbf{\Omega}})$,

(A_{13}) $(|(\sigma_s)_v| + |(\sigma_s)_\theta| + |(\sigma_s)_{vv}| + |(\sigma_s)_{v\theta}| + |(\sigma_s)_{\theta\theta}| + |(\sigma_s)_{vvv}| + |(\sigma_s)_{vv\theta}| + |(\sigma_s)_{v\theta\theta}|$
$+ |(\sigma_s)_{\theta\theta\theta}| + |(\sigma_s)_{\theta\theta\theta\theta}| + |(\sigma_s)_{v\theta\theta\theta}| + |(\sigma_s)_{vv\theta\theta}| + |(\sigma_s)_{vvv\theta}| + |(\sigma_s)_{vvvv}|)(v, \theta; \epsilon)$
$\leq c_{13}|\omega| \mathcal{M}(\epsilon, \overrightarrow{\mathbf{\Omega}})$

where constants s, q, α satisfy $s \in [0, 1]$, $q \geq 1 + s$, $0 \leq \alpha \leq 2s + 1$, the numbers c^j $(j = 1,\ldots,13)$ are positive constants and the nonnegative functions f, g, h, k, l, $\mathcal{M}$ are such that, for some such that, for some $\gamma > 0$,

$$f,\ g,\ h,\ k,\ l,\ \mathcal{M} \in L^{1+\gamma}(\mathbb{R}^+ \times S^{n-1}) \bigcap L^\infty(\mathbb{R}^+ \times S^{n-1}).$$

We assume that the viscosity coefficient μ is a positive constant. In the following, we denote

$$\mathcal{I}(x, t) := \int_0^{+\infty} d\epsilon \int_{S^{n-1}} I(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) d\overrightarrow{\mathbf{\Omega}}$$

for the integrated radiative intensity. In particular,

$$\mathcal{I}(x, 0) = \mathcal{I}_0 = \int_0^{+\infty} d\epsilon \int_{S^{n-1}} I(x, 0; \epsilon, \overrightarrow{\mathbf{\Omega}}) d\overrightarrow{\mathbf{\Omega}}.$$

We define some classes of functions as

$$\begin{aligned}\mathcal{H}^1 = \{(v, u, \theta, \mathcal{I}) \in H^1[0,\ L] \times H_0^1[0,\ L] \times H^1[0,\ L] \times H^2[0,\ L] : v(x) > 0,\ \theta(x) > 0, \\ u(0) = u(L) = 0,\ \mathcal{I}(0) = 0 \text{ for } \omega \in (0, 1) \text{ and } \mathcal{I}(L) = 0 \text{ for } \omega \in (-1, 0)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^i = \{(v, u, \theta, \mathcal{I}) \in H^i[0,\ L] \times H^i[0,\ L] \times H^i[0,\ L] \times H^{i+1}[0,\ L] : v(x) > 0,\ \theta(x) > 0, \\ u(0) = u(L) = 0,\ \theta'(0) = \theta'(L) = 0,\ \mathcal{I}(0) = 0 \text{ for } \omega \in (0, 1) \\ \text{and } \mathcal{I}(L) = 0 \text{ for } \omega \in (-1, 0)\},\quad i = 2, 4,\end{aligned}$$

which become the metric spaces when equipped with the metrics induced from the usual norms.

We are now in a position to state our main theorems.

Theorem 5.2.1. *Assume that the initial data* $(v_0, u_0, \theta_0, \mathcal{I}_0) \in \mathcal{H}^1$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t), u(t), \theta(t), \mathcal{I}(t)) \in L^\infty([0, +\infty), \mathcal{H}^1)$ *to problem (5.2.20)–(5.2.23) with initial boundary value conditions (5.2.14)–(5.2.16) verifying for all* $(x,\ t) \in [0,\ L] \times [0, +\infty)$,

$$0 < C_1^{-1} \le v(x,t) \le C_1, \quad 0 < C_1^{-1} \le \theta(x,t) \le C_1, \tag{5.2.24}$$

and

$$\begin{aligned}&\|v(t)-\overline{v}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^1}^2+\|\mathcal{I}(t)\|_{H^2}^2+\|r(t)-\overline{r}\|_{H^2}^2+\|r_t(t)\|_{H^1}^2\\&\quad+\int_0^t\Big(\|v-\overline{v}\|_{H^1}^2+\|u\|_{H^2}^2+\left\|\theta-\overline{\theta}\right\|_{H^2}^2+\|r-\overline{r}\|_{H^2}^2+\|r_t\|_{H^2}^2+\|\theta_t\|^2\Big)(\tau)\,d\tau\\&\quad+\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}}I_t^2\,d\overrightarrow{\Omega}\,d\epsilon dxd\tau\le C_1.\end{aligned}\tag{5.2.25}$$

Moreover, we have, as $t \to +\infty$,

$$\|v(t)-\overline{v}\|_{H^1}\to 0,\ \ \|u(t)\|_{H^1}\to 0,\ \ \left\|\theta(t)-\overline{\theta}\right\|_{H^1}\to 0,\ \ \|\mathcal{I}(t)\|_{H^2}\to 0,\tag{5.2.26}$$

where constant $\overline{\theta} > 0$ *is determined by* $e(\overline{v},\overline{\theta}) = \int_0^L \left(\frac{1}{2}u_0^2 + e(v_0,\theta_0) + r_0^{1-n}F_R(0)\right)dx$, $\overline{v} = \overline{v_0} = L^{-1}\int_0^L v_0(x)dx$ *and* $\overline{r} = (a^n + n\overline{v}x)^{\frac{1}{n}}$.

Theorem 5.2.2. *Assume that the initial data* $(v_0, u_0, \theta_0, \mathcal{I}_0) \in \mathcal{H}^2$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t), u(t), \theta(t), \mathcal{I}(t)) \in L^\infty([0,+\infty), \mathcal{H}^2)$ *to problem (5.2.20)–(5.2.23) with initial boundary value conditions (5.2.14)–(5.2.16) satisfying for all* $(x,t) \in [0,L] \times [0,+\infty)$,

$$\begin{aligned}&\|v(t)-\overline{v}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^2}^2+\|\mathcal{I}(t)\|_{H^3}^2+\|r(t)-\overline{r}\|_{H^3}^2+\|r_t(t)\|_{H^2}^2\\&\quad+\|u_t(t)\|^2+\|\theta_t(t)\|^2+\int_0^t\Big(\|v-\overline{v}\|_{H^2}^2+\|u\|_{H^3}^2+\left\|\theta-\overline{\theta}\right\|_{H^3}^2+\|r-\overline{r}\|_{H^3}^2\\&\quad+\|r_t\|_{H^3}^2+\|u_{xt}\|^2+\|\theta_{xt}\|^2\Big)(\tau)\,d\tau\le C_2,\end{aligned}\tag{5.2.27}$$

where $\overline{v}, \overline{\theta} > 0$ *and* $\overline{r}$ *are also the same as those of theorem 5.2.1.*

Moreover, we have, as $t \to +\infty$,

$$\|v(t)-\overline{v}\|_{H^2}\to 0,\ \ \|u(t)\|_{H^2}\to 0,\ \ \left\|\theta(t)-\overline{\theta}\right\|_{H^2}\to 0,\ \ \|\mathcal{I}(t)\|_{H^3}\to 0.\tag{5.2.28}$$

Theorem 5.2.3. *Assume that the initial data* $(v_0, u_0, \theta_0, \mathcal{I}_0) \in \mathcal{H}^4$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t), u(t), \theta(t), \mathcal{I}(t)) \in L^\infty([0,+\infty), \mathcal{H}^4)$ *to problem (5.2.20)–(5.2.23) with initial boundary value conditions (5.2.14)–(5.2.16) verifying for all* $(x,t) \in [0,L] \times [0,+\infty)$,

$$\begin{aligned}&\|v(t)-\overline{v}\|_{H^4}^2+\|u(t)\|_{H^4}^2+\left\|\theta(t)-\overline{\theta}\right\|_{H^4}^2+\|\mathcal{I}(t)\|_{H^5}^2+\|r(t)-\overline{r}\|_{H^5}^2+\|r_t(t)\|_{H^4}^2\\&\quad+\|u_t(t)\|_{H^2}^2+\|\theta_t(t)\|_{H^2}^2+\int_0^t\Big(\|v-\overline{v}\|_{H^4}^2+\|u\|_{H^5}^2+\left\|\theta-\overline{\theta}\right\|_{H^5}^2+\|r-\overline{r}\|_{H^5}^2\\&\quad+\|r_t\|_{H^5}^2+\|u_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2+\|u_{tt}\|_{H^1}^2+\|\theta_{tt}\|_{H^1}^2\Big)(\tau)\,d\tau\le C_4,\end{aligned}\tag{5.2.29}$$

where $\overline{v}, \overline{\theta} > 0$ *and* $\overline{r}$ *are also the same as those of theorem 5.2.1.*

Moreover, we have, as $t \to +\infty$,

$$\|v(t)-\overline{v}\|_{H^4} \to 0, \ \ \|u(t)\|_{H^4} \to 0, \ \ \left\|\theta(t)-\overline{\theta}\right\|_{H^4} \to 0, \ \ \|\mathcal{I}(t)\|_{H^5} \to 0. \tag{5.2.30}$$

In what follows, letters C_i $(i=1,\ 2,\ 4)$ will denote the universal positive constants depending on the norms of initial datum $(u_0, v_0, \theta_0, \mathcal{I}_0)$ in $\mathcal{H}^i$ (see below the definitions of $\mathcal{H}^i$) but being independent of t, respectively.

In this chapter, we mainly make full use of the embedding theorem and interpolation technique to establish the global existence, uniqueness and asymptotic behavior of spherically symmetric solutions for the compressible viscous gas with radiation. These results in this chapter are adopted from [119].

5.3 Global Existence and Asymptotic Behavior in $\mathcal{H}^1$

In this section, we establish the global existence and asymptotic behavior in $\mathcal{H}^1$ for the generalized solutions in $\mathcal{H}^1$ to problem (5.2.20)–(5.2.23) with initial boundary value conditions (5.2.14)–(5.2.16) and then complete the proof of theorem 5.2.1 in terms of a series of lemmas.

5.3.1 Global Existence in $\mathcal{H}^1$

At first, we give some basic relations and the fundamental entropy-type energy estimate.

Lemma 5.3.1. *Under assumptions in theorem 5.2.1, there exists a constant* $C_1 > 0$ such that the following estimates hold

$$L^{-1}\int_0^L v(x,t)\,dx = L^{-1}\int_0^L v_0(x)\,dx := \overline{v_0}, \ \ \forall t > 0, \tag{5.3.1}$$

$$\theta(x,t) > 0, \ \ \forall (x,\ t) \in [0,\ L] \times [0,\ +\infty), \tag{5.3.2}$$

$$\int_0^L \left(\theta + \theta^{1+s}\right)(x,t)\,dx \le C_1, \ \ \forall t > 0, \tag{5.3.3}$$

$$U(t) + \int_0^t V(\tau)\,d\tau \le C_1, \ \ \forall t > 0, \tag{5.3.4}$$

where

$$\begin{cases} U(t) = \displaystyle\int_0^L \big((\theta - \log\theta - 1) + \theta^{1+s} + u^2\big)(x,t)\,dx, & (5.3.5)\\ V(t) = \displaystyle\int_0^L \left(\frac{u_x^2}{v\theta} + \frac{(1+\theta^q)\theta_x^2}{v\theta^2}\right)(x,t)\,dx. & (5.3.6)\end{cases}$$

Proof. We can easily get (5.3.1) by integrating (5.2.20) over $Q_t{:} = (0,\, L) \times (0,\, t)$ and noting the boundary conditions. Equation (5.2.22) can be written as

$$e_\theta\theta_t + \theta P_\theta(r^{n-1}u)_x = \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x + \delta\frac{(r^{n-1}u)_x^2}{v} - 2\mu(n-1)(r^{n-2}u^2)_x - v(S_E)_R. \tag{5.3.7}$$

We can derive (5.3.2) from the maximal value principle and the positivity of θ_0.

Multiplying (5.2.21) by u, adding the resultant to (5.2.22), and then integrating it over Q_t and using the boundary conditions, we derive

$$\int_0^L \left(e + \frac{1}{2}u^2\right)dx + \int_0^t\int_0^L v(S_E)_R\,dxd\tau = \int_0^L \left(e_0 + \frac{1}{2}u_0^2\right)dx. \tag{5.3.8}$$

From the definitions of F_R and $(S_E)_R$, we can derive from (5.2.23) that

$$v(S_E)_R = r^{n-1}(F_R)_x \tag{5.3.9}$$

which, together with (5.2.15), leads to

$$\begin{aligned}\int_0^L r^{n-1}(F_R)_x dx &= (r^{n-1}F_R)|_{x=0}^{x=L} - (n-1)\int_0^L \frac{v}{r}F_R dx\\ &= \int_0^{+\infty}\int_{S^{n-1}\cap\{\omega\in(0,1)\}} \omega(b^{n-1}I(L,t;\epsilon,\overrightarrow{\mathbf{\Omega}}) - (n-1)\int_0^L \frac{v}{r}I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})dx)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\\ &\quad - \int_0^{+\infty}\int_{S^{n-1}\cap\{\omega\in(-1,0)\}} \omega(a^{n-1}I(0,t;\epsilon,\overrightarrow{\mathbf{\Omega}}) + (n-1)\int_0^L \frac{v}{r}I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})dx)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon.\end{aligned}$$

Thus, the contribution of the radiation term reads

$$\int_0^t\int_0^L v(S_E)_R dxd\tau = \int_0^t\int_0^L r^{n-1}(F_R)_x dxd\tau \ge 0,$$

which, along with (5.3.8), implies

$$\int_0^L \left(e + \frac{1}{2}u^2\right)dx \le C_1. \tag{5.3.10}$$

Combining (5.3.10) with (5.2.19) and assumptions (A_1)–(A_5) yield (5.3.3).

Similarly to the proof of lemma 3.3.2 (see also lemma 2.1 in Qin *et al.* [106]), estimate (5.3.4) can be shown, thus we omit it. The proof is now complete. ☐

Remark 5.3.1. *By the Jensen inequality, the mean value theorem and (5.3.4), we can know that there exists a point $a(t) \in [0, L]$ and two positive constants α_1, α_2 such that*

$$0 < \alpha_1 \le \tilde{\theta}(t) := L^{-1} \int_0^L \theta(x, t)\, dx = \theta(a(t), t) \le \alpha_2 \tag{5.3.11}$$

where α_1, α_2 are two roots of the equation $y - \log y - 1 = C_1$.

Let

$$\sigma(x, t) = \delta \frac{(r^{n-1}u)_x}{v} - P,$$
$$\phi(x, t) = \int_0^t \sigma(x, \tau)\, d\tau + \int_0^x (r_0^{1-n} u_0)(y)\, dy + (n-1) \int_0^t \int_x^L (r^{-n}u^2)(y, \tau)\, dy d\tau,$$

then, noting that $r_t = u$, we can derive from (5.2.20), by the mean value theorem, that there exists a point $x_0(t) \in [0, L]$ for any $t \ge 0$ such that

$$\int_0^L v\phi\, dx = \phi(x_0(t), t) \int_0^L v dx := v^* \phi(x_0(t), t),$$

i.e.,

$$v^* = \frac{1}{\phi(x_0(t), t)} \int_0^L v\phi\, dx. \tag{5.3.12}$$

Lemma 5.3.2. *For any $t \ge 0$, we have the following expression*

$$v(x, t) = \frac{D(x, t)}{B(x, t)} \left(1 + \delta^{-1} \int_0^t \frac{B(x, \tau) v(x, \tau) P(x, \tau)}{D(x, \tau)} d\tau\right) \tag{5.3.13}$$

where

$$B(x, t) = \exp\Big[\delta^{-1}\Big(\frac{1}{v^*} \int_0^t \int_0^L \Big(\frac{u^2}{n} + Pv\Big)(x, \tau)\, dx d\tau + \frac{(n-1)a^n}{nv^*} \int_0^t \int_0^L (r^{-n}u^2)(x, \tau)\, dx d\tau$$
$$+ (n-1) \int_0^t \int_x^L (r^{-n}u^2)(y, \tau)\, dy d\tau\Big)\Big],$$
$$D(x, t) = v_0(x) \exp\left[\delta^{-1}\left(\frac{1}{v^*} \int_0^L v_0(x) \int_0^x (r_0^{1-n}u_0)(y)\, dy dx - \int_0^x (r_0^{1-n}u_0)(y)\, dy\right.\right.$$
$$\left.\left. + \int_{x_0(t)}^x (r^{1-n}u)(y, t)\, dy\right)\right].$$

Proof. Similarly to the proof of lemma 2.3.2, we can easily prove this lemma. See also lemma 4.1.8 in Qin [99]. ☐

Lemma 5.3.3. *Under assumptions in theorem 5.2.1, the following estimate holds for all* $(x,t)\in[0,\ L]\times[0,+\infty)$,

$$0<C_1^{-1}\le v(x,t)\le C_1. \tag{5.3.14}$$

Proof. The main idea of the proof is similar to that of lemma 2.3.3. See also lemma 5.2.4 in Qin and Huang [109]. Here we omit it. □

Lemma 5.3.4. *Under assumptions in theorem 5.2.1, the following estimates hold for any* $t>0$,

$$\int_0^t \|u(\tau)\|_{L^\infty}^2\,d\tau\le C_1, \tag{5.3.15}$$

$$\|v_x(t)\|^2+\int_0^t\int_0^L\left(1+\theta^{1+s}\right)v_x^2(x,\tau)\,dxd\tau\le C_1, \tag{5.3.16}$$

$$\|u(t)\|^2+\int_0^t\|u_x(\tau)\|^2\,d\tau\le C_1. \tag{5.3.17}$$

Proof. Similarly to lemmas 3.3.1–3.3.6, we can complete the proof. See also lemma 2.3 of Qin *et al.* [106] and lemma 2.4 in Qin *et al.* [106]. Thus we omit the details. □

The next lemma is concerned with an estimate on radiative density I, which plays an important role in our subsequent analysis.

Lemma 5.3.5. *Under assumptions in theorem 5.2.1, the following estimate holds for any* $t>0$,

$$\begin{aligned}&\int_0^\infty\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv+\int_0^\infty\int_0^L\int_{S^{n-1}} v\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxdv\\&\quad+\int_0^\infty\int_0^L\int_{S^{n-1}} v\sigma_s I^2\,d\overrightarrow{\Omega}\,dxdv\le C_1\int_0^L\theta^{1+\alpha}(x,t)\,dx.\end{aligned} \tag{5.3.18}$$

Proof. Multiplying (5.2.23) by I and then integrating the resultant over $[0,\ L]\times S^{n-1}$ and using the boundary conditions, we obtain

$$\begin{aligned}&\int_{S^{n-1}}\omega(b^{n-1}I^2(L,t;\overrightarrow{\Omega},\epsilon)-a^{n-1}I^2(0,t;\overrightarrow{\Omega},\epsilon))\,d\overrightarrow{\Omega}-(n-1)\int_0^L\int_{S^{n-1}}\omega\frac{v}{r}I^2\,d\overrightarrow{\Omega}\,dx\\&\quad=2\int_0^L\int_{S^{n-1}} vI(\sigma_a(B-I)+\sigma_s(\widetilde{I}-I))\,d\overrightarrow{\Omega}\,dx.\end{aligned} \tag{5.3.19}$$

Noting the boundary condition (5.2.22), we know

$$\begin{aligned}
&\int_0^{+\infty}\int_{S^{n-1}} \omega(b^{n-1}I^2(L,t;\overrightarrow{\Omega},\epsilon) - a^{n-1}I^2(0,t;\overrightarrow{\Omega},\epsilon))\,d\overrightarrow{\Omega}\,d\epsilon\\
&\quad= \int_0^{+\infty}\int_{S^{n-1}\cap\{\omega\in(0,1)\}} \omega b^{n-1}I^2(L,t;\overrightarrow{\Omega},\epsilon)\,d\overrightarrow{\Omega}\,d\epsilon\\
&\qquad - \int_0^{+\infty}\int_{S^{n-1}\cap\{\omega\in(-1,0)\}} \omega a^{n-1}I^2(0,t;\overrightarrow{\Omega},\epsilon)\,d\overrightarrow{\Omega}\,d\epsilon\\
&\quad\geq 0.
\end{aligned} \tag{5.3.20}$$

Integrating (5.3.19) with respect to ϵ over $[0, +\infty)$ and using the Young inequality, lemma 5.3.1, (5.3.27) and the assumption (A_8), we find

$$\begin{aligned}
&\int_0^{\infty}\int_{S^{n-1}} \omega(b^{n-1}I^2(L,t;\overrightarrow{\Omega},v) - a^{n-1}I^2(0,t;\overrightarrow{\Omega},v))\,d\overrightarrow{\Omega}\,dv\\
&\quad + \int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv + \int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxdv\\
&\quad\leq \frac{1}{2}\int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv + C_1\int_0^{\infty}\int_0^L\int_{S^{n-1}} \theta^{1+\alpha}f(v,\overrightarrow{\Omega})\,d\overrightarrow{\Omega}\,dxdv\\
&\quad\leq \frac{1}{2}\int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv + C_1\int_0^L \theta^{1+\alpha}(x,t)\,dx
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv + \int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxdv\\
&\quad\leq C_1\int_0^L \theta^{1+\alpha}(x,t)\,dx.
\end{aligned}$$

Similarly, we also derive from (5.3.19) by the Young inequality and assumptions (A_8) that

$$\int_0^{\infty}\int_0^L\int_{S^{n-1}} v(\sigma_a+\sigma_s)I^2\,d\overrightarrow{\Omega}\,dxdv \leq C_1\int_0^L \theta^{1+\alpha}(x,t)\,dx.$$

Therefore, we complete the proof of (5.3.18). □

Obviously, we can obtain the following result from lemmas 5.3.1 and 5.3.5.

Corollary 5.3.1. *Under assumptions in theorem 5.2.1, the following estimate also holds for any* $t > 0$,

$$\begin{aligned}
&\int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_a I^2\,d\overrightarrow{\Omega}\,dxdv + \int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxdv\\
&\quad + \int_0^{\infty}\int_0^L\int_{S^{n-1}} v\sigma_s I^2\,d\overrightarrow{\Omega}\,dxdv \leq C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-s,0\}}(x,t).
\end{aligned}$$

The following lemma establishes two key inequalities which are important to deal with the boundedness of temperature θ. Here we adopt the technique from lemma 7 in Ducomet and Nečasová [32].

Lemma 5.3.6. *Under assumptions in theorem 5.2.1, the following estimates hold for any* $t>0$,

$$\int_0^L (\theta^2+\theta^{2+2s}+u_x^2)(x,t)\,dx+\int_0^t \|u_x(\tau)\|_{L^\infty}^2\,d\tau\le C_1, \tag{5.3.21}$$

$$\begin{aligned}\int_0^t \|u_{xx}(\tau)\|^2\,d\tau+\int_0^t\int_0^L (1+\theta^{q+s})\theta_x^2(x,\tau)\,dxd\tau\\ +\int_0^t\int_0^L (1+\theta^{2+2s})v_x^2(x,\tau)\,dxd\tau\le C_1.\end{aligned} \tag{5.3.22}$$

Proof. Noting the formula (5.3.11) in remark 5.3.1, we can define the auxiliary function for any $\xi>0$,

$$F(\xi)=\int_{\tilde\theta}^{\xi} e_\theta(v,\tau)\,d\tau.$$

Thus it follows from the assumption (A_1) that $F(\xi)\le c_1|\xi-\tilde\theta|(1+\xi^s)$. Multiplying (5.2.22) by $F(\theta)$ over Q_t, we can derive

$$\begin{aligned}&\int_0^L F(\theta)e\,dx+\int_0^t\int_0^L e_\theta\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_x^2\,dxd\tau\\
&\le\int_0^t\int_0^L |ee_v+PF(\theta)||(r^{n-1}u)_x|\,dxd\tau\\
&\quad+\int_0^t\int_0^L \delta|F(\theta)|\frac{(r^{n-1}u)_x^2}{v}\,dxd\tau+\int_0^t\int_0^L |F(\theta)||(r^{n-2}u^2)_x|\,dxd\tau\\
&\quad+\int_0^t\int_0^L |F(\theta)|v|(S_E)_R|\,dxd\tau.\end{aligned}$$

Noting that

$$\begin{aligned}\theta^{2+2s}(x,t)&\le(\theta^{1+s}(x,t)-\tilde\theta^{1+s})^2+C\tilde\theta^{2+2s}\\
&\le CV_1(t)\int_0^L\theta^{2+2s-q}(x,t)\,dx+C\le CV_1(t)+C,\end{aligned}$$

and using the assumptions (A_1) and (A_5)–(A_6), lemma 5.3.3 and the Sobolev embedding theorem, we have for any $\varepsilon>0$,

$$\begin{aligned}&\int_0^L(\theta^2+\theta^{2+2s})\,dx+\int_0^t\int_0^L(1+\theta^{q+s})\theta_x^2\,dxd\tau\\
&\le C_\varepsilon+\varepsilon\int_0^t(1+V_1(\tau))\|u_{xx}\|^2\,d\tau\\
&\quad+C_\varepsilon\int_0^t\left(\|u\|^2+\|u_x\|^2\right)(\tau)\int_0^L\theta^{2+2s}\,dxd\tau+\int_0^t\int_0^L|F(\theta)|v|(S_E)_R|\,dxd\tau.\end{aligned} \tag{5.3.23}$$

Using the Cauchy–Schwarz inequality and the Sobolev embedding inequality, lemmas 5.3.1 and 5.3.5, and noting that the assumption $\alpha \leq 1 + 2s$, we can obtain

$$\begin{aligned}
&\int_0^t \int_0^L |F(\theta)|v|(S_E)_R| dx d\tau \\
&\quad \leq C_1 + C_1 \int_0^t \int_0^L (\theta - \tilde{\theta})^2 (1 + \theta^{2s}) dx d\tau + C_1 \int_0^t \int_0^L v^2 \theta^{1+\alpha}(x,\tau) dx d\tau \\
&\quad \leq C_1 + C_1 \int_0^t V_1(\tau) \int_0^L \theta^{2+2s} dx d\tau + C_1 \int_0^t \|v(\tau)\|_{L^\infty}^2 \int_0^L \theta^{2+2s} dx d\tau \\
&\quad \leq C_1 + C_1 \int_0^t \left(V_1(\tau) + \|v_x(\tau)\|^2\right) \int_0^L \theta^{2+2s} dx d\tau.
\end{aligned} \tag{5.3.24}$$

Multiplying (5.2.21) by $-u_{xx}$ and integrating by parts over Q_t, using (5.2.18) and lemma 5.3.3, we arrive at

$$\begin{aligned}
&\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \\
&\quad \leq C_1 + C_1 \int_0^t \int_0^L |u_{xx}(P_v v_x + P_\theta \theta_x)| dx d\tau + C_1 \int_0^t \int_0^L |v_x (r^{n-1}u)_x u_{xx}| dx d\tau \\
&\quad \leq C_1 + \varepsilon \int_0^t \int_0^L u_{xx}^2 dx d\tau + C_\varepsilon \int_0^t \int_0^L v_x^2 (r^{n-1}u)_x^2 dx d\tau \\
&\qquad + C_\varepsilon \int_0^t \int_0^L (1 + \theta^{q+s}) \theta_x^2 dx d\tau + C_\varepsilon \int_0^t \int_0^L (1 + \theta^{2+2s}) v_x^2 dx d\tau.
\end{aligned} \tag{5.3.25}$$

Noting that $|(r^{n-1}u)_x| \leq C_1(|u| + |u_x|)$, we derive from lemmas 5.3.1 and 5.3.4 that for any $\varepsilon > 0$,

$$C_\varepsilon \int_0^t \int_0^L v_x^2 (r^{n-1}u)_x^2 dx d\tau \leq C_\varepsilon \varepsilon^{-1} + \varepsilon \int_0^t \int_0^L u_{xx}^2 dx d\tau. \tag{5.3.26}$$

Thus, taking $\varepsilon > 0$ small enough, inserting (5.3.24) into (5.3.23) and (5.3.26) into (5.3.25) and applying the Gronwall inequality, we complete the proof of this lemma. □

Now applying the Sobolev embedding theorem, the Hölder inequality and (5.3.21), we can readily obtain the next corollary.

Corollary 5.3.2. *Under assumptions in theorem 5.2.1, the following estimate holds for any* $(x, t) \in [0, L] \times [0, +\infty)$,

$$C_1^{-1} - C_1 V_1(t) \leq \theta^{2m}(x,t) \leq C_1 + C_1 V_1(t)$$

with $0 \leq m \leq \frac{q+2s+2}{2}$.

Corollary 5.3.3. *Under assumptions in theorem 5.2.1, the following estimate holds for any* $(x, t) \in [0, L] \times [0, +\infty)$,

$$\int_0^t \int_0^L (1+\theta)^{2m} u_x^2 dx d\tau \le C_1$$

with $0 \le m \le \frac{q+2s+2}{2}$.

Proof. It follows from corollary 5.3.1 and (5.3.24) and (5.3.28) that

$$\int_0^t \int_0^L (1+\theta)^{2m} u_x^2 dx d\tau \le C_1 \int_0^t \int_0^L u_x^2 dx d\tau + C_1 \int_0^t \int_0^L V_1(\tau) u_x^2 dx d\tau \le C_1. \qquad \square$$

Using the Young inequality and above lemmas, we easily obtain the following estimate.

Lemma 5.3.7. *Under assumptions in theorem 5.2.1, the following estimate holds for any* $t > 0$,

$$\left\|(r^{n-1}u)_x(t)\right\|^2 + \int_0^t \|u_t(\tau)\|^2 d\tau \le C_1. \tag{5.3.27}$$

Proof. Multiplying (5.2.21) by u_t over Q_t and using the Young inequality, we have for any $\varepsilon > 0$,

$$\begin{aligned}
&\left\|(r^{n-1}u)_x(t)\right\|^2 + \int_0^t \|u_t(\tau)\|^2 d\tau \\
&\quad \le C_1 + \frac{\varepsilon}{2}\int_0^t \|u_t(\tau)\|^2 d\tau + C_\varepsilon \int_0^t \int_0^L v_x^2(u^2 + u_x^2) dx d\tau \\
&\qquad + C_1 \int_0^t \int_0^L |u_t(P_v v_x + P_\theta \theta_x)| dx d\tau \\
&\quad \le C_1 + \varepsilon \int_0^t \|u_t(\tau)\|^2 d\tau + C_\varepsilon \int_0^t \|v_x\|^2 \Big(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2\Big) dx d\tau \\
&\qquad + C_\varepsilon \int_0^t \int_0^L (1+\theta^{2+2s}) v_x^2 dx d\tau + C_\varepsilon \int_0^t \int_0^L (1+\theta^{q+s}) \theta_x^2 dx d\tau
\end{aligned}$$

which, by taking $\varepsilon > 0$ small enough, along with lemmas 5.3.4 and 5.3.6, leads to (5.3.27). $\square$

The next lemma is devoted to dealing with estimates on temperature θ.

Lemma 5.3.8. *Under assumptions in theorem 5.2.1, the following estimates hold that for any* $t > 0$,

$$\int_0^L (1+\theta)^{2q} \theta_x^2 dx + \int_0^t \int_0^L (1+\theta)^{q+s} \theta_t^2 dx d\tau \le C_1, \tag{5.3.28}$$

$$\|\theta(x,\tau)\|_{L^\infty} \le C_1. \tag{5.3.29}$$

Proof. Let

$$K(v,\theta) = \int_0^\theta \frac{r^{2n-2}\kappa(v,\xi)}{v}d\xi,$$
$$X(t) = \int_0^t \int_0^L (1+\theta)^{q+s}\theta_t^2 dxd\tau,$$
$$Y(t) = \int_0^L (1+\theta)^{2q}\theta_x^2 dx.$$

Then it is easy to verify that

$$K_t = K_v(r^{n-1}u)_x + \frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t,$$
$$K_{xt} = \left(\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}\right)_t + K_v(r^{n-1}u)_{xx} + K_{vv}v_x(r^{n-1}u)_x + \left(\frac{r^{2n-2}\kappa(v,\theta)}{v}\right)_v v_x\theta_t.$$

However, we easily know from assumptions (A_6) and (A_7) that

$$|K_v| + |K_{vv}| \le C(1+\theta^{1+q}). \tag{5.3.30}$$

We rewrite (5.2.22) as

$$e_\theta\theta_t + \theta P_\theta(r^{n-1}u)_x - \delta\frac{(r^{n-1}u)_x^2}{v} = \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x - 2\mu(n-1)(r^{n-2}u^2)_x - v(S_E)_R. \tag{5.3.31}$$

Multiplying (5.3.31) by K_t and integrating the resultant over Q_t, we easily obtain

$$\int_0^t\int_0^L \left(e_\theta\theta_t + \theta P_\theta(r^{n-1}u)_x - \delta\frac{(r^{n-1}u)_x^2}{v}\right)K_t dxd\tau + \int_0^t\int_0^L \frac{r^{2n-2}\kappa\theta_x}{v}K_{tx}dxd\tau$$
$$+ 2\mu(n-1)\int_0^t\int_0^L (r^{n-2}u^2)_x K_t dxd\tau + \int_0^t\int_0^L v(S_E)_R K_t dxd\tau = 0. \tag{5.3.32}$$

Now we use lemmas 5.3.1–5.3.7 to estimate each term in (5.3.32). Obviously,

$$\int_0^t\int_0^L e_\theta\theta_t^2\frac{r^{2n-2}\kappa(v,\theta)}{v}dxd\tau \ge C_1^{-1}X(t), \tag{5.3.33}$$

$$\left|\int_0^t\int_0^L e_\theta\theta_t K_v(r^{n-1}u)_x dxd\tau\right| \le \frac{1}{8}X(t) + C_1\left(1 + \sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{2+s+q}\right) \tag{5.3.34}$$

and applying the Cauchy inequality, we can derive

$$\begin{aligned}
&\left|\int_0^t\int_0^L \Big(\theta P_\theta (r^{n-1}u)_x - \delta\frac{(r^{n-1}u)_x^2}{v}\Big)\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t dxd\tau\right| \\
&\quad\le \frac{1}{8}X(t) + C_1\int_0^t\int_0^L \Big((1+\theta)^{q+s+2}|(r^{n-1}u)_x|^2 + (1+\theta)^{q-s}|(r^{n-1}u)_x|^4\Big)dxd\tau \\
&\quad\le \frac{1}{8}X(t) + C_1\int_0^t\int_0^L \Big((1+\theta)^{q+s+2}(|u|^2+|u_x|^2) + (1+\theta)^{q-s}(|u|^4+|u_x|^4)\Big)dxd\tau \\
&\quad\le \frac{1}{8}X(t) + C_1\int_0^t\int_0^L (1+\theta)^{q-s}(|u|^4+|u_x|^4)dxd\tau. \qquad (5.3.35)
\end{aligned}$$

However,

$$\begin{aligned}
\int_0^t\int_0^L (1+\theta)^{q-s}u^4 dxd\tau &\le \Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q-s}\Big)\int_0^t \|u\|_{L^\infty}^2\int_0^L u^2 dxd\tau \\
&\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q-s}\Big), \qquad (5.3.36)
\end{aligned}$$

and applying corollaries 5.3.1, 5.3.2 and lemma 5.3.6, we derive for $q_1 = \max\{\frac{q}{2}-2s-1, 0\}$,

$$\begin{aligned}
&\int_0^t\int_0^L (1+\theta)^{q-s}u_x^4 dxd\tau \\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q_1}\Big)\int_0^t\int_0^L (1+\theta)^{\frac{q+2s+2}{2}}u_x^4 dxd\tau \\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q_1}\Big)\Big(\int_0^t\int_0^L u_x^4 dxd\tau + \int_0^t V_1^{\frac{1}{2}}(\tau)\int_0^L u_x^4 dxd\tau\Big) \\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q_1}\Big)\Big(\int_0^t \|u_x\|^3\|u_{xx}\|d\tau + \int_0^t V_1^{\frac{1}{2}}(\tau)\|u_x\|^3\|u_{xx}\|d\tau\Big) \\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q_1}\Big)\Big(\sup_{0\le\tau\le t}\|u_x(\tau)\|^2\Big(\int_0^t \|u_x(\tau)\|^2 d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|u_{xx}(\tau)\|^2 d\tau\Big)^{\frac{1}{2}} \\
&\qquad + \sup_{0\le\tau\le t}\|u_x(\tau)\|^3\Big(\int_0^t V_1(\tau)d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|u_{xx}(\tau)\|^2 d\tau\Big)^{\frac{1}{2}}\Big) \\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q_1}\Big), \qquad (5.3.37)
\end{aligned}$$

which, along with (5.3.35) and (5.3.36), gives

$$\begin{aligned}
&\int_0^t\int_0^L \Big(\theta P_\theta (r^{n-1}u)_x - \epsilon\frac{(r^{n-1}u)_x^2}{v}\Big)\frac{r^{2n-2}\kappa(v,\theta)}{v}\theta_t dxd\tau \\
&\quad\le \frac{1}{8}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}^{q-s}\Big). \qquad (5.3.38)
\end{aligned}$$

It follows from lemma 5.3.6 and corollaries 5.3.1, 5.3.2 that

$$\begin{aligned}
&\left|\int_0^t\int_0^L\left(\theta P_\theta(r^{n-1}u)_x-\delta\frac{(r^{n-1}u)_x^2}{v}\right)K_v(r^{n-1}u)_x dxd\tau\right|\\
&\quad\le C_1\int_0^t\int_0^L\Big((1+\theta)^{q+s+2}|(r^{n-1}u)_x|^2+(1+\theta)^{q+1}|(r^{n-1}u)_x|^3\Big)dxd\tau\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^{q+s+2}|(r^{n-1}u)_x|^2dxd\tau+C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+1}\\
&\qquad\times\int_0^t\int_0^L(|u|^3+|u_x|^3)dxd\tau\\
&\quad\le C_1+C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+1}\Big(\int_0^t\|u\|_{L^\infty}^2\int_0^L|u|dxd\tau+\int_0^t\int_0^L|u_x|^3dxd\tau\Big)\\
&\quad\le C_1+C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+1}\int_0^t\|u_x(\tau)\|^{\frac{5}{2}}\|u_{xx}(\tau)\|^{\frac{1}{2}}d\tau\\
&\quad\le C_1+C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+1}\sup_{0\le\tau\le t}\|u_x(\tau)\|\Big(\int_0^t\|u_x(\tau)\|^2d\tau\Big)^{\frac{3}{4}}\\
&\qquad\times\Big(\int_0^t\|u_{xx}(\tau)\|^2d\tau\Big)^{\frac{1}{4}}\\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+1}.
\end{aligned}\tag{5.3.39}$$

Now let us consider the various contributions in the second integral of (5.3.32). By lemmas 5.3.1–5.3.7 and corollary 5.3.3, we have

$$\int_0^t\int_0^L\frac{r^{2n-2}\kappa\theta_x}{v}\left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_t dxd\tau\ge C_1^{-1}Y(t)-C_1, \tag{5.3.40}$$

$$\begin{aligned}
&\left|\int_0^t\int_0^L\frac{r^{2n-2}\kappa\theta_x}{v}K_v(r^{n-1}u)_{xx}dxd\tau\right|\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^{1+2q}|\theta_x(r^{n-1}u)_{xx}|dxd\tau\\
&\quad\le C_1\Big(\int_0^t\int_0^L(1+\theta)^{s+q}\theta_x^2dxd\tau\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_0^t\int_0^L(1+\theta)^{q-s+2}|(r^{n-1}u)_{xx}|^2dxd\tau\Big)^{\frac{1}{2}}\\
&\quad\le C_1\Big(\int_0^t\int_0^L(1+\theta)^{3q-s+2}(|u|^2+|u_x|^2+|u_{xx}|^2)dxd\tau\Big)^{\frac{1}{2}}\\
&\quad\le C_1(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty})^{\frac{3q-s+2}{2}}
\end{aligned}\tag{5.3.41}$$

and

$$\begin{aligned}
&\left|\int_0^t\int_0^L \frac{r^{2n-2}\kappa\theta_x}{v}K_{vv}v_x(r^{n-1}u)_x dxd\tau\right|\\
&\quad\le C_1\int_0^t\int_0^L(1+\theta)^{1+2q}|\theta_x v_x(r^{n-1}u)_x|dxd\tau\\
&\quad\le C_1\Big(\int_0^t\int_0^L(1+\theta)^{s+q}\theta_x^2 dxd\tau\Big)^{\frac12}\Big(\int_0^t\int_0^L(1+\theta)^{3q-s+2}v_x^2|(r^{n-1}u)_x|^2 dxd\tau\Big)^{\frac12}\\
&\quad\le C_1(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty})^{\frac{3q-s+2}{2}}\Big(\int_0^t\int_0^L v_x^2(r^{n-1}u)_x^2 dxd\tau\Big)^{\frac12}\\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\frac{3q-s+2}{2}}\Big(\int_0^t\int_0^L v_x^2u^2 dxd\tau+\int_0^t\int_0^L v_x^2u_x^2 dxd\tau\Big)^{\frac12}\\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\frac{3q-s+2}{2}}\Big(\int_0^t\|u\|_{L^\infty}^2\|v_x\|^2 d\tau+\int_0^t\|u_x\|_{L^\infty}^2\|v_x\|^2 d\tau\Big)^{\frac12}\\
&\quad\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\frac{3q-s+2}{2}}. \qquad (5.3.42)
\end{aligned}$$

Noting the following facts

$$\begin{aligned}
\int_0^t\int_0^L(r^{n-1}u)_x^4 dxd\tau &\le C_1\int_0^t\int_0^L(u^4+u_x^4)dxd\tau\\
&\le C_1+C_1\int_0^t\|u_x(\tau)\|^3\|u_{xx}(\tau)\|d\tau\\
&\le C_1+C_1\sup_{0\le\tau\le t}\|u_x(\tau)\|^2\Big(\int_0^t\|u_x(\tau)\|^2d\tau\Big)^{\frac12}\Big(\int_0^t\|u_{xx}(\tau)\|^2d\tau\Big)^{\frac12}\\
&\le C_1,
\end{aligned}$$

$$\begin{aligned}
\int_0^t\int_0^L(r^{n-2}u^2)_x^2 dxd\tau &\le C_1\int_0^t\int_0^L(u^4+u^2u_x^2)dxd\tau\\
&\le C_1+C_1\int_0^t\|u_x\|_{L^\infty}^2 d\tau\le C_1,
\end{aligned}$$

$$\begin{aligned}
\int_0^t\big\|v(S_E)_R\big\|^2 d\tau &\le \int_0^t\int_0^{+\infty}\int_0^L\int_{S^{n-1}}(v^2\sigma_a^2I^2+v^2\sigma_a^2B^2+v^2\sigma_s^2(\widetilde{I}-I)^2)d\overrightarrow{\Omega}\,dxd\epsilon d\tau\\
&\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{\alpha-s,0\}}
\end{aligned}$$

and from equation (5.2.22), it follows

$$
\begin{aligned}
&\int_0^t \Big\|\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_x\Big\|^2 d\tau \\
&\le C_1\int_0^t (\|e_\theta\theta_t\|^2 + \big\|\theta P_\theta(r^{n-1}u)_x\big\|^2 + \big\|(r^{n-1}u)_x^2\big\|^2 + \big\|(r^{n-2}u^2)_x\big\|^2 + \big\|v(S_E)_R\big\|^2)d\tau \\
&\le C_1\int_0^t\int_0^L ((1+\theta)^{2s}\theta_t^2 + (1+\theta)^{2s+2}u^2 + (1+\theta)^{2s+2}u_x^2 + (r^{n-1}u)_x^4 + (r^{n-2}u^2)_x^2)\,dxd\tau \\
&\quad + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{\alpha-s,0\}} \\
&\le C_1\int_0^t\int_0^L (1+\theta)^{q+s}\theta_t^2\,dxd\tau + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{\alpha-s,0\}}. \qquad (5.3.43)
\end{aligned}
$$

Thus, by the Sobolev inequality, lemmas 5.3.4 and 5.3.7, we can conclude

$$
\begin{aligned}
&\Big|\int_0^t\int_0^L \frac{r^{2n-2}\kappa\theta_x}{v}\Big(\frac{r^{2n-2}\kappa}{v}\Big)_v v_x\theta_t\,dxd\tau\Big| \\
&\le C_1\int_0^t\int_0^L (1+\theta)^q\Big|\theta_x v_x\theta_t\frac{r^{2n-2}\kappa\theta_x}{v}\Big|dxd\tau \\
&\le \frac{1}{16}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q-s} \\
&\quad\times\Big(\int_0^t\int_0^L (1+\theta)^{2q}\theta_x^2\,dxd\tau + \int_0^t\int_0^L (1+\theta)^q|\theta_x|\Big|\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_x\Big|dxd\tau\Big) \\
&\le \frac{1}{16}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{2q-s+2}\int_0^t V_1(\tau)\,d\tau \\
&\quad + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q-s}\Big(\int_0^t V_1(\tau)\,d\tau\Big)^{\frac12}\Big(\int_0^t\int_0^L (1+\theta)^q\theta^2\Big|\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_x\Big|^2 dxd\tau\Big)^{\frac12} \\
&\le \frac{1}{16}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{2q-s+2} + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q-\frac{s}{2}+1} \\
&\quad\times\Big(\int_0^t\int_0^L (1+\theta)^{q-s}\Big|\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_x\Big|^2 dxd\tau\Big)^{\frac12} \\
&\le \frac{1}{8}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{2q-s+2} + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\frac{3(q-s)+\alpha+2}{2}}.
\end{aligned}
\qquad (5.3.44)
$$

By lemma 5.3.4 and corollary 5.3.3, using the Cauchy inequality, we have

$$
\Big|\int_0^t\int_0^L (r^{n-2}u^2)_x\Big(K_v(r^{n-1}u)_x + \frac{r^{2n-2}\kappa}{v}\theta_t\Big)dxd\tau\Big| \le \frac{1}{8}X(t) + C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q-s}. \qquad (5.3.45)
$$

The last contribution in (5.3.32) can be estimated as follows,

$$\begin{aligned}
&\left|\int_0^t\int_0^L v(S_E)_R K_t dxd\tau\right| \\
&\leq \int_0^t\int_0^L\left|\int_0^{+\infty}\int_{S^{n-1}} v\sigma_a B d\overrightarrow{\boldsymbol{\Omega}}d\epsilon\right||K_t|dxd\tau \\
&\quad+\int_0^t\int_0^L\left|\int_0^{+\infty}\int_{S^{n-1}} v\sigma_a I d\overrightarrow{\boldsymbol{\Omega}}d\epsilon\right||K_t|dxd\tau \\
&\quad+\int_0^t\int_0^L\left|\int_0^{+\infty}\int_{S^{n-1}} v\sigma_s(\widetilde{I}-I) d\overrightarrow{\boldsymbol{\Omega}}d\epsilon\right||K_t|dxd\tau \\
&=: M_1+M_2+M_3. \qquad (5.3.46)
\end{aligned}$$

It follows from corollaries 5.3.1–5.3.3 that

$$\begin{aligned}
M_1 &\leq C_1\int_0^t\int_0^L(1+\theta^\alpha)|K_t|dxd\tau \\
&\leq C_1\int_0^t\int_0^L(1+\theta^{q+\alpha+1})|(r^{n-1}u)_x|dxd\tau + C_1\int_0^t\int_0^L(1+\theta^{q+\alpha})|\theta_t|dxd\tau \\
&\leq \frac{1}{8}X(t)+C_1\Big(1+\sup_{0\leq\tau\leq t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{q+2\alpha-s}. \qquad (5.3.47)
\end{aligned}$$

Using the assumption (A_9), the Cauchy inequality and corollary 5.3.1, we have

$$\begin{aligned}
M_2 &\leq C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\left|v\sigma_a I\Big(K_v(r^{n-1}u)_x+\frac{r^{2n-2}\kappa}{v}\theta_t\Big)\right|d\overrightarrow{\boldsymbol{\Omega}}d\epsilon dxd\tau \\
&\leq C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} v\sigma_a I^2 d\overrightarrow{\boldsymbol{\Omega}}d\epsilon dxd\tau + C_1\int_0^t\int_0^L(1+\theta^{2q+2})|(r^{n-1}u)_x|^2dxd\tau \\
&\quad+C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}}(1+\theta^q)|I\theta_t|d\overrightarrow{\boldsymbol{\Omega}}d\epsilon dxd\tau \\
&\leq \frac{1}{8}X(t)+C_1\Big(1+\sup_{0\leq\tau\leq t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{q-s,\alpha-s\}}. \qquad (5.3.48)
\end{aligned}$$

Using the same technique, we also get

$$M_3\leq\frac{1}{8}X(t)+C_1\Big(1+\sup_{0\leq\tau\leq t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{q-s,\alpha-s\}}. \qquad (5.3.49)$$

Inserting estimates (5.3.47)–(5.3.49) into (5.3.46), we get

$$\left|\int_0^t\int_0^L v(S_E)_R K_t dxd\tau\right|\leq\frac{1}{8}X(t)+C_1\Big(1+\sup_{0\leq\tau\leq t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{q-s,q+2\alpha-s\}}. \qquad (5.3.50)$$

Inserting all previous estimates (5.3.33)–(5.3.35), (5.3.38)–(5.3.42), (5.3.44), (5.3.45) and (5.3.50) into (5.3.32), we obtain

$$X(t)+Y(t)\le C_1\Big(1+\sup_{0\le\tau\le t}\|\theta(x,\tau)\|_{L^\infty}\Big)^{\max\{q+s+2,q+2\alpha-s,\frac{3}{2}(q-s)+\alpha-s,2q-s+2,\frac{3q-s+2}{2}\}}. \tag{5.3.51}$$

By lemmas 5.3.1, 5.3.4, 5.3.6 and the Hölder inequality, there exists a point $b(t)\in[0,\,L]$ such that for any $t>0$,

$$\begin{aligned}\theta^{q+s+2}(x,t)-\theta^{q+s+2}(b(t),t)&=\int_{b(t)}^{x}(\theta^{q+s+2}(x,t))_x\,dx\le C_1\int_0^L\theta^{q+s+1}|\theta_x|\,dx\\&\le C_1\Big(\int_0^L\theta^{2s+2}\,dx\Big)^{\frac12}Y^{\frac12}(t).\end{aligned}$$

Thus we get

$$\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\le C_1+C_1\,Y^{\frac{1}{2(q+s+2)}}(t).$$

Using assumptions on q, s and α, we easily know that $\max\{q+s+2,q+2\alpha-s,\frac{3}{2}(q-s)+\alpha-s,2q-s+2,\frac{3q-s+2}{2}\}<2(q+s+2)$. Thus, by the Young inequality and (5.3.51), it follows that

$$X(t)+Y(t)\le C_1$$

which yields

$$\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\le C_1.$$

Therefore, we can complete the proof. □

The next two lemmas are aimed at coping with some estimates on radiative density I.

Lemma 5.3.9. *The following estimates hold that for any* $t>0$,

$$\int_0^{+\infty}\int_{S^{n-1}}I(x,t;\epsilon,\overrightarrow{\Omega})\,d\overrightarrow{\Omega}\,d\epsilon\le C_1, \tag{5.3.52}$$

$$\Big\|\int_0^{+\infty}\int_{S^{n-1}}I\,d\overrightarrow{\Omega}\,d\epsilon\Big\|_{L^\infty(Q_t)}+\Big\|\int_0^{+\infty}\int_{S^{n-1}}\widetilde{I}\,d\overrightarrow{\Omega}\,d\epsilon\Big\|_{L^\infty(Q_t)}\le C_1, \tag{5.3.53}$$

$$\Big\|\int_0^{+\infty}\int_{S^{n-1}}I^2\,d\overrightarrow{\Omega}\,d\epsilon\Big\|_{L^\infty(Q_t)}+\Big\|\int_0^{+\infty}\int_{S^{n-1}}|I_x|\,d\overrightarrow{\Omega}\,d\epsilon\Big\|_{L^\infty(Q_t)}\le C_1, \tag{5.3.54}$$

$$\left\|\int_0^{+\infty}\int_{S^{n-1}}|I_t|\,d\overrightarrow{\Omega}\,d\epsilon\right\|_{L^\infty(Q_t)}+\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}}I_t^2(x,\tau;\epsilon,\overrightarrow{\Omega})\,d\overrightarrow{\Omega}\,d\epsilon dxd\tau\le C_1. \tag{5.3.55}$$

Proof. We consider the following integro-differential equation

$$\begin{cases} r^{n-1}\omega I_x=vS=v(\sigma_aB+\sigma_s\widetilde{I})-v(\sigma_a+\sigma_s)I,\\ I(0,t;\epsilon,\overrightarrow{\Omega})=0 \ \text{ for } \ \omega\in(0,1),\\ I(L,t;\epsilon,\overrightarrow{\Omega})=0 \ \text{ for } \ \omega\in(-1,0),\\ I(x,0;\epsilon,\overrightarrow{\Omega})=I_0(x,\epsilon,\overrightarrow{\Omega}). \end{cases} \tag{5.3.56}$$

Solving the ordinary differential equation and using the boundary conditions, we arrive at

$$I(x,t;\epsilon,\overrightarrow{\Omega})=\begin{cases} \displaystyle\int_0^x\exp\Big(\int_x^y\frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)\frac{v}{r^{n-1}\omega}(\sigma_aB+\sigma_s\widetilde{I})\,dy, & \text{for all } \omega\in(0,1),\\ \displaystyle-\int_x^L\exp\Big(\int_x^y\frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)\frac{v}{r^{n-1}\omega}(\sigma_aB+\sigma_s\widetilde{I})\,dy, & \text{for all } \omega\in(-1,0). \end{cases} \tag{5.3.57}$$

Using the Young inequality and lemmas 5.3.5 and 5.3.8, we have for all $\omega\in(0,1)$,

$$\begin{aligned} \int_0^{+\infty}\int_{S^{n-1}}I\,d\overrightarrow{\Omega}\,d\epsilon &=\int_0^{+\infty}\int_{S^{n-1}}\int_0^x\exp\Big(\int_x^y\frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)\frac{v}{r^{n-1}\omega}(\sigma_aB+\sigma_s\widetilde{I})\,dyd\overrightarrow{\Omega}\,d\epsilon\\ &\le\left|\int_0^{+\infty}\int_{S^{n-1}}\int_0^L\frac{v}{r^{n-1}\omega}(\sigma_aB+\sigma_s\widetilde{I})\,dxd\overrightarrow{\Omega}\,d\epsilon\right|\\ &\le\left|\int_0^L vr^{1-n}\int_0^{+\infty}\int_{S^{n-1}}\frac{\sigma_aB}{\omega}d\overrightarrow{\Omega}\,d\epsilon dx\right|\\ &\quad+\left|\int_0^L vr^{1-n}\int_0^{+\infty}\int_{S^{n-1}}\frac{\sigma_s\widetilde{I}}{\omega}d\overrightarrow{\Omega}\,d\epsilon dx\right|\\ &\le C_1\int_0^L\theta^{1+\alpha}dx\\ &\quad+C_1\left|\int_0^L vr^{1-n}\int_0^{+\infty}\int_{S^{n-1}}\sigma_s\Big(\frac{1}{\omega^2}+(\widetilde{I}-I)^2+I^2\Big)d\overrightarrow{\Omega}\,d\epsilon dx\right|\\ &\le C_1. \end{aligned} \tag{5.3.58}$$

Analogously, we have the same result for all $\omega\in(-1,\ 0)$. Thus, using lemma 5.3.8, we get

$$\left\|\int_0^{+\infty}\int_{S^{n-1}} I\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right\|_{L^\infty(Q_t)} \le C_1. \tag{5.3.59}$$

Furthermore, we also get

$$\left\|\int_0^{+\infty}\int_{S^{n-1}} \widetilde{I}\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right\|_{L^\infty(Q_t)} \le C_1. \tag{5.3.60}$$

Similarly to (5.3.58), using the Young and the Hölder inequalities, we derive

$$\begin{aligned}\int_0^{+\infty}\int_{S^{n-1}} I^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon &\le \int_0^{+\infty}\int_{S^{n-1}}\left(\int_0^L \frac{v}{r^{n-1}\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dy\right)^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\\ &\le C_1\int_0^{+\infty}\int_{S^{n-1}}\left(\int_0^L \frac{\sigma_a v^2}{r^{2n-2}\omega^2}dx\cdot\int_0^L \sigma_a B^2\,dx\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\\ &\quad + C_1\int_0^{+\infty}\int_{S^{n-1}}\left(\int_0^L \frac{\sigma_s v^2}{r^{2n-2}\omega^2}dx\cdot\int_0^L \sigma_s\widetilde{I}^2\,dx\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\\ &\le C_1+C_1\int_0^{+\infty}\int_{S^{n-1}}\int_0^L \sigma_s(\widetilde{I}-I)^2+I^2)\,dxd\overrightarrow{\mathbf{\Omega}}\,d\epsilon\le C_1\end{aligned} \tag{5.3.61}$$

which implies

$$\left\|\int_0^{+\infty}\int_{S^{n-1}} I^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right\|_{L^\infty(Q_t)} \le C_1. \tag{5.3.62}$$

It follows from (5.2.23) that

$$I_x = -\frac{r^{1-n}v}{\omega}(\sigma_a+\sigma_s)I+\frac{r^{1-n}v}{\omega}(\sigma_a B+\sigma_s\widetilde{I}).$$

Integrating the above equality and using assumptions (A_8)–(A_{11}), we can derive

$$\begin{aligned}\int_0^{+\infty}\int_{S^{n-1}}|I_x|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon &\le C_1\int_0^{+\infty}\int_{S^{n-1}}\frac{(\sigma_a+\sigma_s)|I|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon + C_1\int_0^{+\infty}\int_{S^{n-1}}\frac{(\sigma_a|B|+\sigma_s|\widetilde{I}|)}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\\ &\le C_1+C_1\int_0^{+\infty}\int_{S^{n-1}}|I|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon + C_1\int_0^{+\infty}\int_{S^{n-1}}|\widetilde{I}|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\end{aligned}$$

which, along with (5.3.59) and (5.3.60), leads to

$$\left\|\int_0^{+\infty}\int_{S^{n-1}} |I_x|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right\|_{L^\infty(Q_t)} \le C_1. \tag{5.3.63}$$

It follows from (5.3.57) that for any $\omega \in (0,\, 1)$,

$$\begin{aligned}
I_t &= \int_0^x \exp\Big(\int_x^y \frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)\Big(\int_x^y \frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)_t \frac{v}{r^{n-1}\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dy\\
&\quad+\int_0^x \exp\Big(\int_x^y \frac{v(\sigma_a+\sigma_s)}{r^{n-1}\omega}dz\Big)\frac{v}{r^{n-1}\omega}(\sigma_a B+\sigma_s\widetilde{I})\Big(\frac{v}{r^{n-1}\omega}(\sigma_a B+\sigma_s\widetilde{I})\Big)_t dy\\
&=: N_1+N_2. \qquad (5.3.64)
\end{aligned}$$

Using the Young inequality, lemmas 5.3.1–5.3.8 and the assumptions (A_9)–(A_{11}) and (A_{13}), we have

$$\begin{aligned}
&\int_0^t\int_0^{+\infty}\int_{S^{n-1}} N_1^2\, d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\le C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\Bigg(\int_0^x\int_x^y\Big(\frac{1}{\omega}((r^{1-n}v)_t(\sigma_a+\sigma_s)+r^{1-n}v(\sigma_a+\sigma_s)_v v_t\\
&\quad+r^{1-n}v(\sigma_a+\sigma_s)_\theta\theta_t)\Big)\,dz\cdot\frac{v(\sigma_a B+\sigma_s\widetilde{I})}{r^{n-1}\omega}\,dy\Bigg)^2 d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\le C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\Big(\int_0^L\Big(\frac{(r^{n-1}u)_x^2}{\omega^2}(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_v\Big)^2\\
&\quad+\frac{\theta_t^2}{\omega^2}(\sigma_a+\sigma_s)_\theta^2)\,dx\Big)\int_0^L\frac{(r^{1-n}v)^2}{\omega^2}(\sigma_a^2B^2+\sigma_s^2\widetilde{I}^2)\,dx d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\le C_1\int_0^t\int_0^L\Big((r^{n-1}u)_x^2+\theta_t^2\Big)\,dx d\tau+C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\int_0^L\widetilde{I}^2\,dx d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\le C_1 \qquad (5.3.65)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\int_0^{+\infty}\int_{S^{n-1}} N_2^2\, d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\quad\le C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\Big(\int_0^x\Big(\frac{1}{\omega}((r^{1-n}v)_t(\sigma_a B+\sigma_s\widetilde{I})+r^{1-n}v((\sigma_a)_v v_t B\\
&\qquad+(\sigma_a)_\theta\theta_t B+\sigma_a B_\theta\theta_t+(\sigma_s)_v v_t\widetilde{I}+(\sigma_s)_\theta\theta_t\widetilde{I}+\sigma_s\widetilde{I}_t\Big)\,dy\Big)^2 d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\quad\le C_1\int_0^t\int_0^L((r^{n-1}u)_x^2+\theta_t^2)\,dx d\tau+C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\int_0^x I_t^2\,dy d\overrightarrow{\Omega}\, d\epsilon\, d\tau\\
&\quad\le C_1+C_1\int_0^t\int_0^{+\infty}\int_{S^{n-1}}\int_0^x I_t^2\,dy d\overrightarrow{\Omega}\, d\epsilon\, d\tau \qquad (5.3.66)
\end{aligned}$$

which, together with (5.3.65), implies

$$\int_0^t\int_0^{+\infty}\int_{S^{n-1}} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon\, d\tau \le C_1 + C_1\int_0^x\left(\int_0^t\int_0^{+\infty}\int_{S^{n-1}} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon\, d\tau\right)dy. \tag{5.3.67}$$

Using the same technique, we have the above inequality for any $\omega \in (-1, 0)$. By the Gronwall inequality for any fixed $t > 0$, we have

$$\int_0^t\int_0^{+\infty}\int_{S^{n-1}} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon\, d\tau \le C_1 e^{C_1 x} \le C_1 e^{C_1 L} \le C_1. \tag{5.3.68}$$

Similarly, we can obtain

$$\int_0^{+\infty}\int_{S^{n-1}} |I_t|\, d\overrightarrow{\Omega}\, d\epsilon \le C_1. \tag{5.3.69}$$

Thus, the estimate (5.3.55) follows from (5.3.68) and (5.3.69). □

Lemma 5.3.10. *Under assumptions in theorem 5.2.1, the following estimates hold that for any* $t > 0$,

$$\|\mathcal{I}_x(t)\| \le C_1, \quad \|\mathcal{I}_{xx}(t)\| \le C_1. \tag{5.3.70}$$

Proof. By lemma 5.3.9, we have

$$\begin{aligned}
\|\mathcal{I}_x(t)\|^2 &= \int_0^L\left(\int_0^{+\infty}\int_{S^{n-1}} I_x\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\
&\le \int_0^L\left(\int_0^{+\infty}\int_{S^{n-1}} \frac{r^{1-n}v}{|\omega|}((\sigma_a+\sigma_s)|I| + \sigma_a B + \sigma_s|\widetilde{I}|)\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\
&\le C_1 + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^{n-1}} |I|\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\
&\quad + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^{n-1}} |\widetilde{I}|\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\
&\le C_1.
\end{aligned} \tag{5.3.71}$$

By virtue of the direct computation, we also have

$$\begin{aligned}
\|\mathcal{I}_{xx}(t)\|^2 &= \int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}((r^{1-n}v)_x S + r^{1-n}vS_x)\, d\overrightarrow{\Omega}\, d\epsilon\Big)^2 dx\\
&\le C_1\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{(r^{1-n}v)_x}{\omega} S\, d\overrightarrow{\Omega}\, d\epsilon\Big)^2 dx\\
&\quad + C_1\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{r^{1-n}v}{\omega} S_x\, d\overrightarrow{\Omega}\, d\epsilon\Big)^2 dx\\
&=: G_1 + G_2.
\end{aligned} \tag{5.3.72}$$

Using lemma 5.3.9, we see that

$$\begin{aligned} G_1 &= \int_0^L (r^{1-n}v)_x^2 \Big(\int_0^{+\infty} \int_{S^{n-1}} \frac{1}{\omega}(\sigma_a(B-I)+\sigma_s(\widetilde{I}-I))\, d\vec{\Omega}\, d\epsilon \Big)^2 dx \\ &\le C_1 \int_0^L (r^{1-n}v)_x^2 dx \le C_1. \end{aligned} \tag{5.3.73}$$

Similarly,

$$\begin{aligned} G_2 = &\int_0^L \Big(\int_0^{+\infty} \int_{S^{n-1}} \frac{r^{1-n}v}{\omega}\big(\big((\sigma_a)_v v_x + (\sigma_a)_\theta \theta_x\big)(B-I) + \sigma_a(B_\theta\theta_x - I_x) \\ &+ \big((\sigma_s)_v v_x + (\sigma_s)_\theta \theta_x\big)(\widetilde{I}-I) + \sigma_s(\widetilde{I}-I)_x\big)\, d\vec{\Omega}\, d\epsilon\Big)^2 dx \\ \le\, & C_1 + C_1 \int_0^L (v_x^2+\theta_x^2)\, dx \le C_1. \end{aligned} \tag{5.3.74}$$

Inserting (5.3.73), (5.3.74) into (5.3.72), we obtain the desired estimate. □

5.3.2 Asymptotic Behavior in $\mathcal{H}^1$

In this subsection, we mainly establish the next two lemmas to show the asymptotic behavior of solutions to the problem (5.2.20)–(5.2.23) with the initial boundary conditions (5.2.14)–(5.2.16) in $\mathcal{H}^1$.

Lemma 5.3.11. *Under assumptions in theorem 5.2.1, we have, as $t \to +\infty$,*

$$\|v(t) - \overline{v}\|_{H^1} \to 0, \tag{5.3.75}$$

$$\|u(t)\|_{H^1} \to 0, \tag{5.3.76}$$

$$\big\|\theta(t) - \overline{\theta}\big\|_{H^1} \to 0, \quad \big\|\theta(t) - \overline{\theta}\big\|_{L^\infty} \to 0 \tag{5.3.77}$$

and for all $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < C_1^{-1} \le \theta(x,t) \le C_1. \tag{5.3.78}$$

Proof. By lemmas 5.3.4, 5.3.6 and 5.3.7, we can know

$$\int_0^t \left(\|v_x\|^2 + \|u_x\|^2\right)(\tau)\, d\tau \le C_1 \tag{5.3.79}$$

and

$$\int_0^t \left(\Big|\frac{d}{dt}\|v_x\|^2\Big| + \Big|\frac{d}{dt}\|u_x\|^2\Big|\right)(\tau)\, d\tau \le C_1 \tag{5.3.80}$$

which lead to (5.3.75) and (5.3.76).

Multiplying (5.3.31) by $e_\theta^{-1}\theta_{xx}$, integrating the resultant over $(0, L)$ and using the Young inequality, the interpolation inequality and lemmas 5.3.1–5.3.4 and 5.3.6–5.3.8, we can conclude for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\|\theta_x\|^2 + 2\int_0^L \frac{r^{2n-2}\kappa}{e_\theta v}\theta_{xx}^2 dx \\
&\quad = 2\int_0^L e_\theta^{-1}\left(\theta P_\theta (r^{n-1}u)_x - \delta\frac{(r^{n-1}u)_x^2}{v} - \left(\frac{r^{2n-2}\kappa}{v}\right)_x \theta_x \right. \\
&\qquad \left. + 2\mu(n-1)(r^{n-2}u^2)_x + v(S_E)_R\right)\theta_{xx}\, dx \\
&\quad \le \frac{\varepsilon}{2}\|\theta_{xx}\|^2 + C_1\Big(\|(r^{n-1}u)_x\|^2 + \|(r^{n-1}u)_x\|_{L^4}^2 + \|v_x\theta_x\|^2 \\
&\qquad + \|\theta_x\|^2 + \|\theta_x\|_{L^4}^2 + \|(S_E)_R\|^2\Big) \\
&\quad \le \varepsilon\|\theta_{xx}\|^2 + C_1\Big(\|u_x\|^2 + \|u_{xx}\|^2 + \|\theta_x\|^2 + \|(S_E)_R\|^2\Big). \qquad (5.3.81)
\end{aligned}$$

Now we need to estimate the term $\|(S_E)_R\|^2$ in (5.3.81) and derive from the Young and the Hölder inequalities that

$$\begin{aligned}
\|(S_E)_R\|^2 &= \int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} (\sigma_a(B-I) + \sigma_s(\widetilde{I}-I))\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \sigma_a(B-I)\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx \\
&\quad + C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \sigma_s(\widetilde{I}-I)\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \sigma_a\, d\overrightarrow{\Omega}\, d\epsilon \int_0^{+\infty}\int_{S^{n-1}} \sigma_a(B^2+I^2)\, d\overrightarrow{\Omega}\, d\epsilon\right) dx \\
&\quad + C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \sigma_s\, d\overrightarrow{\Omega}\, d\epsilon \int_0^{+\infty}\int_{S^{n-1}} \sigma_s(\widetilde{I}-I)^2\, d\overrightarrow{\Omega}\, d\epsilon\right) dx \\
&\le C_1\int_0^L \theta^{1+\alpha} dx + C_1 \le C_1 \qquad (5.3.82)
\end{aligned}$$

which, together with (5.3.81), leads to

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(\tau)\|^2 d\tau \le C_1. \qquad (5.3.83)$$

Meanwhile, taking $\varepsilon > 0$ small enough and using lemmas 5.3.1–5.3.9, we derive from (5.3.81) that

$$\frac{d}{dt}\|\theta_x\|^2 + C_1^{-1}\int_0^L (1+\theta^{q-s})\theta_{xx}^2 dx \le C_1(1+\|u_{xx}\|^2) \qquad (5.3.84)$$

which, along with theorem 1.2.10, yields

$$\lim_{t\to+\infty}\|\theta_x(t)\|^2=0. \tag{5.3.85}$$

By the Poincaré inequality, we derive

$$\left\|\theta(t)-\overline{\theta}\right\|_{H^1}\leq C_1\|\theta_x(t)\|$$

which, combined with (5.3.85), gives (5.3.77).

Similarly to the proof of (2.3.110) in lemma 2.3.12, we can obtain (5.3.78) from (5.3.31), (5.3.77) and lemmas 5.3.1–5.3.8. □

The next lemma concerns the asymptotic behavior of the radiative intensity and can be proved similarly to lemma 3.4 in Qin *et al.* [106].

Lemma 5.3.12. *Under assumptions in theorem 5.2.1, we have as* $t\to+\infty$,

$$\|\mathcal{I}(t)\|_{H^2}\to 0. \tag{5.3.86}$$

Proof. The direct computation yields from (5.2.23) and lemma 5.3.9

$$\begin{aligned}\frac{d}{dt}\|\mathcal{I}_x\|^2&=\frac{d}{dt}\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}}I_x\,d\overrightarrow{\Omega}\,d\epsilon\Big)^2dx\\&=2\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}}I_x\,d\overrightarrow{\Omega}\,d\epsilon\Big)\Big(\int_0^{+\infty}\int_{S^{n-1}}I_{xt}\,d\overrightarrow{\Omega}\,d\epsilon\Big)dx\\&\leq C_1\Big|\int_0^L\int_0^{+\infty}\int_{S^{n-1}}I_{xt}\,d\overrightarrow{\Omega}\,d\epsilon\,dx\Big|\\&\leq C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}\big|r^{1-n}(r^{n-1}u)_xS\\&\quad+(1-n)r^{-n}uvS+r^{1-n}vS_t\big|\,d\overrightarrow{\Omega}\,d\epsilon\,dx\\&=:H_1+H_2+H_3.\end{aligned} \tag{5.3.87}$$

Now we estimate each term in the above inequality by assumptions (A_8)–(A_{13}) and lemmas 5.3.7–5.3.11. Obviously, noting the expression of S and using assumptions (A_9)–(A_{11}), the Cauchy inequality and lemma 5.3.9, we have

$$\begin{aligned}H_1&=C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}|r^{1-n}(r^{n-1}u)_xS|\,d\overrightarrow{\Omega}\,d\epsilon\,dx\\&=C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}\big|r^{1-n}(r^{n-1}u)_x(\sigma_a(B-I)+\sigma_s(\tilde{I}-I))\big|\,d\overrightarrow{\Omega}\,d\epsilon\,dx\\&\leq C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}|(r^{n-1}u)_x|(|\sigma_a(B-I)|+|\sigma_s(\tilde{I}-I)|)\,d\overrightarrow{\Omega}\,d\epsilon\,dx\\&\leq C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}|(r^{n-1}u)_x|\frac{1}{|\omega|}(c_{10}|\omega|h+c_9|\omega|^2g|I|+c_{11}|\omega|^2k|\tilde{I}-I|)\,d\overrightarrow{\Omega}\,d\epsilon\,dx\\&\leq C_1\int_0^L u_x^2\,dx+C_1.\end{aligned} \tag{5.3.88}$$

Similarly,

$$\begin{aligned}
H_2 &= C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}|(1-n)r^{-n}uvS|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}\big|uv(\sigma_a(B-I)+\sigma_s(\tilde I-I))\big|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}|uv|(|\sigma_a(B-I)|+|\sigma_s(\tilde I-I)|)\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}|u|\frac{1}{|\omega|}(c_8|\omega|\theta^{1+\alpha}f+c_9|\omega|^2g|vI|+c_{11}|\omega|^2k|v\tilde I-I)|)\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L u^2(1+v^2)\,dx+C_1\le C_1. \qquad (5.3.89)
\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}
H_3 &= C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{|\omega|}|r^{1-n}vS_t|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{|v|}{|\omega|}\big|(\sigma_a(B-I)+\sigma_s(\tilde I-I))_t\big|\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{|v|}{|\omega|}(|((\sigma_a)_v v_t+(\sigma_a)_\theta\theta_t)(B-I)|+|\sigma_a(B_\theta\theta_t-I_t)|\\
&\quad+|((\sigma_s)_v v_t+(\sigma_s)_\theta\theta_t)(\tilde I-I)|+|\sigma_s(\tilde I-I)_t|)\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&=: \sum_{j=1}^{9}H_{3j}. \qquad (5.3.90)
\end{aligned}$$

Using assumption (A_{10}) and the Young inequality, we have

$$\begin{aligned}
H_{31} &= C_1\int_0^L|v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_a)_v v_tB|}{|\omega|}\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L|v(r^{n-1}u)_x|\int_0^{+\infty}\int_{S^{n-1}}\frac{c_{10}|\omega|h}{|\omega|}\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L|v(r^{n-1}u)_x|\,dx\le C_1\int_0^L u_x^2\,dx+C_1. \qquad (5.3.91)
\end{aligned}$$

Analogously, using assumptions (A_8)–(A_{13}) and the Young inequality, we also have

$$\begin{aligned}
H_{32} &= C_1\int_0^L|v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_a)_v v_tI|}{|\omega|}\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\
&\le C_1\int_0^L u_x^2\,dx+C_1, \qquad (5.3.92)
\end{aligned}$$

$$\begin{aligned} H_{33} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_a)_\theta\theta_t B|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L \theta_t^2 dx + C_1, \end{aligned} \tag{5.3.93}$$

$$\begin{aligned} H_{34} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_a)_\theta\theta_t I|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L \theta_t^2 dx + C_1, \end{aligned} \tag{5.3.94}$$

$$\begin{aligned} H_{35} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|\sigma_a B_\theta\theta_t|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L \theta_t^2 dx + C_1, \end{aligned} \tag{5.3.95}$$

$$\begin{aligned} H_{36} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|\sigma_a I_t|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_t^2\, d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx + C_1 \\ &\le C_1, \end{aligned} \tag{5.3.96}$$

$$\begin{aligned} H_{37} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_s)_v v_t(\tilde{I}-I)|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L u_x^2 dx + C_1, \end{aligned} \tag{5.3.97}$$

$$\begin{aligned} H_{38} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|(\sigma_s)_\theta\theta_t(\tilde{I}-I)|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L \theta_t^2 dx + C_1, \end{aligned} \tag{5.3.98}$$

$$\begin{aligned} H_{39} &= C_1\int_0^L |v|\int_0^{+\infty}\int_{S^{n-1}}\frac{|\sigma_s(\tilde{I}-I)_t|}{|\omega|}d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx \\ &\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_t^2\, d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx + C_1 \\ &\le C_1. \end{aligned} \tag{5.3.99}$$

Inserting all previous estimates (5.3.88) and (5.3.99) into (5.3.87), we obtain

$$\frac{d}{dt}\|\mathcal{I}_x(t)\|^2 \le C_1\Big(\|u_x\|^2 + \|\theta_t\|^2\Big) + C_1 \tag{5.3.100}$$

which, together with the Shen–Zheng inequality (theorem 1.2.10), yields, as $t \to +\infty$,

$$\|\mathcal{I}_x(t)\| \to 0. \tag{5.3.101}$$

By the direct computation, it also follows from (5.2.23) and lemma 5.3.10 that

$$\|\mathcal{I}_{xx}\|^2 \le C_1\Big(\|v_x\|^2 + \|\theta_x\|^2 + \|\mathcal{I}_x\|^2\Big). \tag{5.3.102}$$

Then using (5.3.101) and lemma 5.3.11, we get as $t \to +\infty$,

$$\|\mathcal{I}_{xx}(t)\| \to 0, \tag{5.3.103}$$

which, combining with (5.3.101), leads to (5.3.86). □

Till now we have completed the proof of theorem 5.2.1. □

5.4 Global Existence and Asymptotic Behavior in $\mathcal{H}^2$

In this section, we shall show theorem 5.2.2, that is, the global existence and asymptotic behavior of solutions in $\mathcal{H}^2$ to the problem (5.2.20)–(5.2.23) with the initial boundary conditions (5.2.14)–(5.2.16) under some relative assumptions.

5.4.1 Global Existence in $\mathcal{H}^2$

We first establish the estimates on u_{xx} and θ_{xx}.

Lemma 5.4.1. *Under assumptions of theorem 5.2.2, there holds that for any $t > 0$,*

$$\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x(t)\|^2_{L^\infty} + \int_0^t \Big(\|u_{xt}\|^2 + \|u_{xx}\|^2\Big)(\tau)\,d\tau \le C_2, \tag{5.4.1}$$

$$\|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|\theta_x(t)\|^2_{L^\infty} + \int_0^t \Big(\|\theta_{xt}\|^2 + \|\theta_{xx}\|^2\Big)(\tau)\,d\tau \le C_2. \tag{5.4.2}$$

Proof. Similarly to the proof of (2.4.3) in lemma 2.3.1, we can obtain estimate (5.4.1).

Differentiating (5.2.22) with respect to t and multiplying the result by θ_t over $[0, L]$, we have that for any $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\|\sqrt{e_\theta}\theta_t\|^2 + C_1^{-1}\|\theta_{xt}\|^2\\
&\quad\leq \varepsilon\|\theta_{xt}\|^2 + C_1(\|\theta_x\|^2 + \|\theta_t\|^2 + \|\theta_t\|_{L^3}^3 + \big\|(r^{n-1}u)_x\big\|^2 + \big\|(r^{n-1}u)_{xt}\big\|^2\\
&\qquad + \Big(\|\theta_t\| + \|\theta_t\|^{\frac{1}{2}}\|\theta_{tx}\|^{\frac{1}{2}})\|\theta_{xt}\| + \big\|(v(S_E)_R)_t\big\|^2\Big).
\end{aligned}
\tag{5.4.3}
$$

Integrating (5.4.3) with respect to t and using lemmas 5.3.1–5.3.8 and the Young inequality, we derive for any $\varepsilon > 0$,

$$
\begin{aligned}
&\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(\tau)\|^2\,d\tau\\
&\quad\leq C_2 + \varepsilon\int_0^t \|\theta_{xt}(\tau)\|^2\,d\tau + C_1\int_0^t \Big(\|\theta_t\|^3 + \|\theta_t\|^{\frac{5}{2}}\|\theta_{tx}\|^{\frac{1}{2}}\Big)(\tau)\,d\tau\\
&\qquad + C_1\int_0^t \big\|\big(v(S_E)_R\big)_t(\tau)\big)\big\|^2\,d\tau\\
&\quad\leq C_2 + \varepsilon\int_0^t \|\theta_{xt}(\tau)\|^2\,d\tau + C_1\sup_{0\leq\tau\leq t}\|\theta_t(\tau)\|^{\frac{4}{3}} + C_1\int_0^t \big\|(v(S_E)_R)_t(\tau)\big\|^2\,d\tau\\
&\quad\leq C_2 + \varepsilon\int_0^t \|\theta_{xt}(\tau)\|^2\,d\tau + C_1\sup_{0\leq\tau\leq t}\|\theta_t(\tau)\|^2 + C_1\int_0^t \big\|v(S_E)_R)_t(\tau)\big\|^2\,d\tau.
\end{aligned}
\tag{5.4.4}
$$

It follows from lemmas 5.3.7–5.3.9 that

$$
\begin{aligned}
&\int_0^t \big\|v(S_E)_R)_t(\tau)\big\|^2\,d\tau\\
&\quad= \int_0^t\int_0^L ((r^{n-1}u)_x(S_E)_R + v((S_E)_R)_t)^2\,dxd\tau\\
&\quad\leq C_1\int_0^t\int_0^L (r^{n-1}u)_x^2\Big(\int_0^{+\infty}\int_{S^{n-1}} (\sigma_a(B-I) + \sigma_s(\widetilde{I}-I))\,d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon\Big)^2\,dxd\tau\\
&\qquad + C_1\int_0^t\int_0^L \Big(\int_0^{+\infty}\int_{S^{n-1}} (((\sigma_a)_v(r^{n-1}u)_x + (\sigma_a)_\theta\theta_t)(B-I) + \sigma_a(B_\theta\theta_t - I_t)\\
&\qquad + ((\sigma_s)_v(r^{n-1}u)_x + (\sigma_s)_\theta\theta_t)(\widetilde{I}-I) + \sigma_s(\widetilde{I}-I)_t)\,d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon\Big)^2\,dxd\tau\\
&\quad\leq C_1\int_0^t \Big(\big\|(r^{n-1}u)_x\big\|^2 + \|\theta_t\|^2\Big)(\tau)\,d\tau + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_t^2\,d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon dxd\tau\\
&\quad\leq C_1.
\end{aligned}
\tag{5.4.5}
$$

Inserting (5.4.5) into (5.4.4) and then taking $\varepsilon > 0$ small enough, we obtain

$$
\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(\tau)\|^2\,d\tau \leq C_2.
\tag{5.4.6}
$$

Using the Gagliardo–Nirenberg interpolation inequality and the Young inequality, we derive from (5.2.22) that

$$\|\theta_{xx}\| \leq C_1\big(\|\theta_t\| + \|\theta_x\| + \big\|(r^{n-1}u)_x\big\| + \|v_x\| + \big\|v(S_E)_R\big\|\big) \leq C_2. \tag{5.4.7}$$

Combining (5.4.6) and (5.4.7), we get (5.4.2). □

The following lemma is concerned with estimate on v_{xx}.

Lemma 5.4.2. *Under assumptions of theorem 5.2.2, the following estimates hold that for any $t > 0$,*

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau \leq C_2, \tag{5.4.8}$$

$$\int_0^t \Big(\|u_{xxx}\|^2 + \|\theta_{xxx}\|^2\Big)(\tau)\, d\tau \leq C_2. \tag{5.4.9}$$

Proof. Similarly to the proof of lemma 2.4.2, we easily obtain (5.4.8).

It follows from (5.2.21) that

$$\|u_{xxx}\| \leq C_2(\|u_{xt}\| + \|u_{xx}\| + \|v_{xx}\| + \|\theta_{xx}\|). \tag{5.4.10}$$

Similarly, we can derive from (5.2.22) that

$$\|\theta_{xxx}\| \leq C_2\big(\|\theta_{xt}\| + \|\theta_{xx}\| + \|u_{xx}\| + \|v_{xx}\| + \big\|(v(S_E)_R)_x\big\|\big). \tag{5.4.11}$$

By the definition of $(S_E)_R$ and lemma 5.3.9, it follows from that

$$\begin{aligned}\int_0^t \big\|\big(v(S_E)_R\big)_x\big\|^2 d\tau &= \int_0^t \int_0^L (v_x(S_E)_R + v((S_E)_R)_x)^2\, dx d\tau \\ &\leq C_1 \int_0^t \|v_x\|^2 + \|\theta_x\|^2)\, d\tau \\ &\quad + C_1 \bigg\| \int_0^{+\infty} \int_{S^{n-1}} |I_x|\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon \bigg\|^2_{L^\infty(Q_t)} \\ &\leq C_1 \end{aligned} \tag{5.4.12}$$

which, together with (5.4.10), (5.4.11), implies (5.4.9). □

The next lemma is aimed at obtaining estimate on $\mathcal{I}_{xxx}$.

Lemma 5.4.3. *Under assumptions of theorem 5.2.2, the following estimate holds for any $t > 0$,*

$$\|\mathcal{I}_{xxx}(t)\| \leq C_2. \tag{5.4.13}$$

Proof. It follows from (5.2.23) and the definitions of I and $\mathcal{I}$ that

$$\begin{aligned}
\|\mathcal{I}_{xxx}\|^2 &= \int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(r^{1-n}vS)_{xx}\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{(r^{1-n}v)_{xx}}{\omega} S\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\quad + C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{(r^{1-n}v)_{x}}{\omega} S_x\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\quad + C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{r^{1-n}v}{\omega} S_{xx}\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{5.4.14}$$

Employing the Gagliardo–Nirenberg interpolation inequality and using lemmas 5.3.9 and 5.4.2, we conclude

$$\begin{aligned}
J_1 &\le C_1\int_0^L (r^{1-n}v)_{xx}^2\left(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(\sigma_a(B-I)+\sigma_s(\mathcal{I}-I))\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L (r^{1-n}v)_{xx}^2\, dx \le C_2,
\end{aligned} \tag{5.4.15}$$

and

$$\begin{aligned}
J_2 &\le C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{(r^{1-n}v)_x}{\omega}(((\sigma_a)_v v_x + (\sigma_a)_\theta\theta_x)(B-I) + \sigma_a(B_\theta\theta_x - I_x)\right. \\
&\qquad \left. + ((\sigma_s)_v v_x + (\sigma_s)_\theta\theta_x)(\mathcal{I}-I) + \sigma_s(\mathcal{I}-I)_x)\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L (v_x^2 + v_x^4 + v_x^2\theta_x^2)\, dx \\
&\le C_1 + C_1\|v_x\|_{L^\infty}^2\left(\|v_x\|^2 + \|\theta_x\|^2\right) \\
&\le C_1 + C_1\left(\|v_x\|\|v_{xx}\| + \|v_x\|^2\right) \le C_2.
\end{aligned} \tag{5.4.16}$$

Similarly, we also derive that

$$\begin{aligned}
J_3 &\le C_1\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} \frac{(r^{1-n}v)}{\omega}(((\sigma_a)_v v_x + (\sigma_a)_\theta\theta_x)(B-I) + \sigma_a(B_\theta\theta_x - I_x)\right. \\
&\qquad \left. + ((\sigma_s)_v v_x + (\sigma_s)_\theta\theta_x)(\mathcal{I}-I) + \sigma_s(\mathcal{I}-I)_x)_x\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dx \\
&\le C_1\int_0^L (v_{xx}^2 + v_x^4 + v_x^2\theta_x^2 + \theta_x^4 + \theta_{xx}^2 + \mathcal{I}_{xx}^2)\, dx \\
&\le C_2 + C_1\left(\|v_x\|\|v_{xx}\| + \|\theta_x\|\|\theta_{xx}\| + \|v_x\|^2 + \|\theta_x\|^2\right)\|\|v_x\|^2 + \|\theta_x\|^2\| \\
&\le C_2,
\end{aligned} \tag{5.4.17}$$

which, along with (5.4.14)–(5.4.16), leads to (5.4.13). □

5.4.2 Asymptotic Behavior in $\mathcal{H}^2$

This subsection concerns the asymptotic behavior of the global solution in $\mathcal{H}^2$.

Lemma 5.4.4. *Under assumptions in theorem 5.2.2, we have, as $t \to +\infty$,*

$$\|v(t) - \overline{v}\|_{H^2} \to 0, \tag{5.4.18}$$

$$\|u(t)\|_{H^2} \to 0, \tag{5.4.19}$$

$$\left\|\theta(t) - \overline{\theta}\right\|_{H^2} \to 0, \tag{5.4.20}$$

$$\|\mathcal{I}(t)\|_{H^3} \to 0. \tag{5.4.21}$$

Proof. Differentiating (5.2.20) with respect to x twice and multiplying the result by v_{xx} in $L^2(0,\, L)$, we have

$$\frac{d}{dt}\|v_{xx}\|^2 \le C_1\left(\|v_{xx}\|^2 + \|u_{xxx}\|^2 + \|u_{xx}\|^2 + \|u_x\|^2\right) \tag{5.4.22}$$

which, together with theorem 5.2.1 and the Gronwall inequality, leads to (5.4.18).

It follows from (5.2.21) that

$$\frac{d}{dt}\|u_t\|^2 + C_1^{-1}\|u_{xt}\|^2 \le C_1\left(\|v_x\|^2 + \|u_x\|^2 + \|u_t\|^2 + \|\theta_t\|^2\right). \tag{5.4.23}$$

Applying theorems 5.2.1 and 1.2.10 to (5.4.23), we get, as $t \to +\infty$,

$$\|u_t(t)\| \to 0 \tag{5.4.24}$$

which gives (5.4.19).

Similarly, we can derive from (5.2.22) that

$$\frac{d}{dt}\|\theta_t\|^2 + C_1^{-1}\|\theta_{xt}\|^2 \le C_2\left(\|\theta_t\|^2 + \|u_x\|^2 + \|u_t\|^2 + \|\theta_x\|^2 + \|u_{xt}\|^2 + \left\|(v(S_E)_R)_t\right\|^2\right)$$

which, along with theorem 1.2.10, lemma 5.4.1 and (5.4.5), gives, as $t \to +\infty$,

$$\|\theta_t(t)\| \to 0. \tag{5.4.25}$$

We can derive from lemma 5.3.9 that

$$\begin{aligned}\frac{d}{dt}\left\|(S_E)_R\right\|^2 &= 2\int_0^L \left(\int_0^{+\infty}\int_{S^{n-1}} S\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)\left(\int_0^{+\infty}\int_{S^{n-1}} S_t\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right) dx\\ &\le C_2\left(1 + \|u_x\|^2 + \|\theta_t\|^2\right)\end{aligned}$$

which, together with (5.3.21), (5.3.27) and theorem 1.2.10, implies, as $t \to +\infty$,

$$\left\|(S_E)_R\right\| \to 0. \tag{5.4.26}$$

By (5.2.22), we see that

$$\|\theta_{xx}\| \le C_2\big(\|\theta_t\| + \|\theta_x\| + \|u_x\| + \|v_x\| + \big\|(S_E)_R\big\|\big), \tag{5.4.27}$$

which, along with (5.3.75), (5.3.76), (5.3.85), (5.4.25) and (5.4.26), gives, as $t \to +\infty$,

$$\|\theta_{xx}(t)\| \to 0. \tag{5.4.28}$$

Thus, as $t \to +\infty$,

$$\big\|\theta(t) - \overline{\theta}\big\|_{H^2} \to 0.$$

Similarly to the proof of lemma 5.3.12 (see also lemma 5.3 in Qin *et al.* [106]), we can obtain from (5.2.23) and lemma 5.4.3 that

$$\|\mathcal{I}_{xxx}\|^2 \le C_1\Big(\|v_x\|^2 + \|v_{xx}\|^2 + \|\theta_{xx}\|^2 + \|\mathcal{I}_{xx}\|^2\Big). \tag{5.4.29}$$

Thus, using (5.4.18)–(5.4.20) and lemmas 5.3.11, 5.3.12, we obtain (5.4.21). □

Till now we have completed the proof of theorem 5.2.2. □

5.5 Global Existence and Asymptotic Behavior in $\mathcal{H}^4$

In this section, we shall prove theorem 5.2.3, that is, the global existence and asymptotic behavior of solutions in $\mathcal{H}^4$ to the problem (5.2.20)–(5.2.23) with the initial boundary conditions (5.2.14)–(5.2.16) under some relative assumptions.

5.5.1 Global Existence in $\mathcal{H}^4$

We begin with estimates on u_{tt} and θ_{tt}.

Lemma 5.5.1. *Under assumptions of theorem 5.2.3, there holds that for any $t > 0$ and $\varepsilon > 0$ small enough,*

$$\|u_{xt}(x,0)\| + \|\theta_{xt}(x,0)\| \le C_4, \tag{5.5.1}$$

$$\|u_{tt}(x,0)\| + \|\theta_{tt}(x,0)\| + \|u_{txx}(x,0)\| + \|\theta_{txx}(x,0)\| \le C_4, \tag{5.5.2}$$

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{ttx}(\tau)\|^2 d\tau \le C_4 + C_4\int_0^t \Big(\|\theta_{txx}\|^2 + \|u_{txx}\|^2\Big)(\tau)\,d\tau, \tag{5.5.3}$$

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{ttx}(\tau)\|^2 d\tau \le{}& C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t \|\theta_{txx}(\tau)\|^2 d\tau \\ &+ C_1\varepsilon\int_0^t \Big(\|u_{ttx}\|^2 + \|u_{txx}\|^2\Big)(\tau)\,d\tau.\end{aligned} \tag{5.5.4}$$

Proof. Differentiating (5.2.21) and (5.2.22) with respect to x, respectively, using theorem 5.2.1 and lemmas 5.4.1 and 5.4.2, we can get

$$\|u_{xt}(t)\| \le C_2\big(\|u_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1}\big), \tag{5.5.5}$$

$$\|\theta_{xt}(t)\| \le C_2\big(\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|\big). \tag{5.5.6}$$

Combining (5.5.5) and (5.5.6), we easily obtain (5.5.1).

Similarly, differentiating (5.2.21) and (5.2.22) with respect to x twice, respectively, using theorem 5.2.1 and lemmas 5.4.1, 5.4.2, we can derive

$$\begin{aligned}\|u_{xxt}(t)\| &\le C_2\big(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_x(t)\|_{L^\infty}\|v_{xxx}(t)\| \\ &\quad + \|v_x(t)\|_{L^\infty}\|u_{xxx}(t)\| + \|u_{xx}(t)\|_{L^\infty}\|v_{xx}(t)\|\big) \\ &\le C_2\big(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2}\big),\end{aligned} \tag{5.5.7}$$

$$\|\theta_{xxt}(t)\| \le C_2\big(\|\theta_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1}\big) \tag{5.5.8}$$

or

$$\|u_{xxxx}(t)\| \le C_2\big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{txx}(t)\|\big), \tag{5.5.9}$$

$$\|\theta_{xxxx}(t)\| \le C_2\big(\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|\theta_{txx}(t)\| + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1}\big). \tag{5.5.10}$$

Differentiating (5.2.21) with respect to t, we have

$$\|u_{tt}(t)\| \le C_2\big(\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|u_{tx}(t)\| + \|u_{txx}(t)\|\big) \tag{5.5.11}$$

$$\le C_2\big(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|\big). \tag{5.5.12}$$

Analogously, we can also derive that

$$\|\theta_{tt}(t)\| \le C_2\big(\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{tx}(t)\| + \|\theta_x(t)\|\|\mathcal{I}_t(t)\|\big) \tag{5.5.13}$$

$$\le C_2\big(\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1} + \|\theta_x(t)\|\|\mathcal{I}_t(t)\|\big). \tag{5.5.14}$$

Thus the estimate (5.5.2) follows from (5.5.7), (5.5.8), (5.5.12) and (5.5.14).

Differentiating (5.2.21) with respect to t twice, multiplying the resultant by u_{tt} in $L^2(0,\ L)$, performing an integration by parts, and using theorem 5.2.2 and the embedding theorem and the Young inequality, we can derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{tt}\|^2 = &-\int_0^L (r^{n-1}u_{tt})_x\Big(\delta\frac{(r^{n-1}u)_x}{v}-P\Big)_{tt}dx-\int_0^L ((r^{n-1})_t u_{tt})_x\Big(\delta\frac{(r^{n-1}u)_x}{v}-P\Big)_t dx\\
&-\int_0^L \Big((r^{n-1})_{tt}u_{tt}+(r^{n-1})_t u_{tt}\Big)_x\Big(\delta\frac{(r^{n-1}u)_x}{v}-P\Big)dx\\
\le &-\int_0^L \delta\frac{r^{2n-2}}{v}u_{ttx}^2\,dx + C_2\big(\|u_{tt}\|+\|u_{xt}u_x\|+\big\|u_x^3\big\|+\|\theta_t u_x\|\\
&+\|u_{xt}\|+\|\theta_{tt}\|+\big\|u_x^2\big\|\big)\|u_{ttx}\|\\
\le &-C_1^{-1}\|u_{ttx}\|^2+C_2\Big(\|u_x\|_{H^1}^2+\|\theta_t\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2+\|u_{tt}\|^2\Big).\end{aligned}\tag{5.5.15}$$

Thus, by theorem 5.2.2,

$$\|u_{tt}(t)\|^2+\int_0^t\|u_{ttx}(\tau)\|^2d\tau\le C_4+C_4\int_0^t\Big(\|u_{tt}\|^2+\|\theta_{tt}\|^2\Big)(\tau)\,d\tau$$

which, along with (5.5.11) and (5.5.13) and lemma 5.3.9, gives estimate (5.5.3).

In the same manner, differentiating (5.2.22) with respect to t twice, multiplying the resultant by θ_{tt} and performing an integration by parts over $L^2(0, L)$, and using the embedding theorem and the Young inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tt}^2\,dx = &-\int_0^L\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_{tt}\theta_{ttx}\,dx-\int_0^L\Big(e_{\theta tt}\theta_t+e_{vtt}(r^{n-1}u)_x\Big)\theta_{tt}\,dx\\
&-\frac{3}{2}\int_0^L e_{\theta t}\theta_{tt}^2\,dx-\int_0^L\Big(e_v+P-\delta\frac{(r^{n-1}u)_x}{v}\Big)(r^{n-1}u)_{xtt}\theta_{tt}\,dx\\
&+\int_0^L\Big(\delta\frac{(r^{n-1}u)_x}{v}-P\Big)_{tt}(r^{n-1}u)_x\theta_{tt}\,dx\\
&-2\int_0^L\Big(e_{vt}+\Big(P-\delta\frac{(r^{n-1}u)_x}{v}\Big)_t\Big)(r^{n-1}u)_{xt}\theta_{tt}\,dx\\
&-2\mu(n-1)\int_0^L(r^{n-2}u^2)_{xtt}\theta_{tt}\,dx-\int_0^L(v(S_E)_R)_{tt}\theta_{tt}\,dx\\
=: &\sum_{i=1}^{8}A_i.\end{aligned}\tag{5.5.16}$$

Similarly to the proof of lemma 2.4.1, by virtue of theorems 5.2.1, 5.2.2 and lemmas 5.4.1–5.4.3, and using the embedding theorem, we derive that for any $\varepsilon\in(0, 1)$,

$$A_1\le -(2C_1)^{-1}\|\theta_{ttx}\|^2+C_2\Big(\|\theta_{xt}\|^2+\|u_{xt}\|^2+\|u_x\|_{H^1}^2+\|\theta_{tt}\|^2\Big),\tag{5.5.17}$$

$$A_2\le\varepsilon\|\theta_{ttx}\|^2+C_2\varepsilon^{-1}\Big(\|u_x\|_{H^1}^2+\|\theta_t\|^2+\|\theta_{xt}\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2\Big),\tag{5.5.18}$$

$$A_3\le C_1(\|\theta_{tt}\|+\|\theta_{ttx}\|)(\|u_x\|+\|\theta_t\|)\|\theta_{tt}\|\le\varepsilon\|\theta_{ttx}\|^2+C_2\varepsilon^{-1}\|\theta_{tt}\|^2,\tag{5.5.19}$$

$$A_4 \le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\|\theta_{tt}\|^2, \tag{5.5.20}$$

$$A_5 \le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\Big(\|\theta_{tt}\|^2 + \|u_x\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{xt}\|^2 + \|u_{xt}\|^2\Big), \tag{5.5.21}$$

$$A_6 \le C_2\|u_{tx}\|^{\frac{1}{2}}\|u_{txx}\|^{\frac{1}{2}}(\|u_x\| + \|\theta_t\| + \|u_{xt}\|)\|\theta_{tt}\|, \tag{5.5.22}$$

$$A_7 \le \varepsilon\|u_{ttx}\|^2 + C_2\varepsilon^{-1}\Big(\|u_x\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{xt}\|^2 + \|\theta_{tt}\|^2\Big). \tag{5.5.23}$$

It follows from (5.5.22) by the Hölder inequality that

$$\begin{aligned}\int_0^t A_6\,d\tau &\le C_2 \sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|\bigg(\int_0^t \|u_{txx}(\tau)\|^2 d\tau\bigg)^{\frac{1}{4}}\bigg(\int_0^t \|u_{tx}(\tau)\|^2 ds\bigg)^{\frac{1}{4}}\\ &\quad\times\bigg(\int_0^t \Big(\|u_x\|^2 + \|\theta_t\|^2 + \|u_{tx}\|^2\Big)(\tau)\,d\tau\bigg)^{\frac{1}{2}}\\ &\le \varepsilon\bigg(\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2 + \int_0^t \|u_{txx}(\tau)\|^2 d\tau\bigg) + C_2\varepsilon^{-3}.\end{aligned} \tag{5.5.24}$$

Now we can estimate A_8 as

$$\begin{aligned}A_8 &\le C_1\int_0^L (r^{n-1}u)_{xt}^2 dx + C_1\big\|(r^{n-1}u)_x\big\|_{L^\infty}^2\int_0^L \big((S_E)_R)\big)_t^2 dx\\ &\quad + C_1\int_0^L \theta_{xx}^2 dx + C_1\int_0^L \big((S_E)_R\big)_{tt}^2 dx\\ &\le C_1\Big(\|u_{xt}\|^2 + \|\theta_{tt}\|^2 + \big\|((S_E)_R)_t\big\|^2\Big) + C_1\int_0^L \big((S_E)_R\big)_{tt}^2 dx.\end{aligned} \tag{5.5.25}$$

By the induction of (5.4.5), we can easily obtain

$$\big\|((S_E)_R)_t\big\|^2 \le C_1\Big(\|u_x\|^2 + \|\theta_t\|^2 + \|\mathcal{I}_t\|^2\Big). \tag{5.5.26}$$

Using the Hölder inequality and the interpolation theorem, we can derive from theorems 5.2.1 and 5.2.2 that

$$\begin{aligned}\int_0^L ((S_E)_R)_{tt}^2 dx &= \int_0^L \bigg(\int_0^{+\infty}\int_{S^{n-1}} (\sigma_a(B - I) + \sigma_s(\mathcal{I} - I)\,d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon\bigg)_{tt}^2 dx\\ &\le C_1\int_0^L ((r^{n-1}u)_x^4 + (r^{n-1}u)_x^2\theta_t^2 + \theta_t^4 + \theta_{tt}^2 + (r^{n-1}u)_{xt}^2)\,dx\\ &\quad + C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{tt}^2 d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon dx\\ &\le C_1\Big(\|u_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\theta_{tt}\|^2 + \|u_{xt}\|^2\Big)\\ &\quad + C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{tt}^2 d\overrightarrow{\boldsymbol{\Omega}}\,d\epsilon dx.\end{aligned} \tag{5.5.27}$$

Differentiating (5.3.57) with respect to t twice and after the lengthy calculation, we can derive

$$\int_0^t\int_0^{+\infty}\int_{S^{n-1}} I_{tt}^2 d\overrightarrow{\Omega}\, d\epsilon\, d\tau \leq C_2 + C_2\int_0^t \|\theta_{tt}(\tau)\|^2 d\tau + C_2\int_0^x\int_0^t\int_0^{+\infty}\int_{S^{n-1}} I_{tt}^2 d\overrightarrow{\Omega}\, d\epsilon\, d\tau\, dy. \tag{5.5.28}$$

Applying the Gronwall inequality to (5.5.28), we can obtain

$$\begin{aligned}\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{tt}^2 d\overrightarrow{\Omega}\, d\epsilon\, dx\, d\tau &\leq \Big(C_2 + C_2\int_0^t \|\theta_{tt}(\tau)\|^2 d\tau\Big) e^{C_2 x}\\ &\leq C_2 + C_2\int_0^t \|\theta_{tt}(\tau)\|^2 d\tau.\end{aligned} \tag{5.5.29}$$

Inserting (5.5.26), (5.5.27) and (5.5.29) into (5.5.25), we have

$$A_8 \leq C_1\Big(\|u_{xt}\|^2 + \|\theta_{tt}\|^2 + \|u_x\|^2 + \|\theta_t\|^2 + \|\mathcal{I}_t\|^2\Big). \tag{5.5.30}$$

Thus we derive from (5.5.16)–(5.5.24) and (5.5.30) that for any $\varepsilon \in (0, 1)$ small enough,

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{ttx}(\tau)\|^2 d\tau &\leq C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t \|\theta_{tt}(\tau)\|^2 d\tau\\ &\quad + C_1\varepsilon\Big(\sup_{0\leq\tau\leq t}\|\theta_{tt}(\tau)\|^2 + \int_0^t \Big(\|u_{txx}\|^2 + \|u_{ttx}\|^2\Big)(\tau)\, d\tau\Big).\end{aligned} \tag{5.5.31}$$

Therefore taking supremum in t on the left-hand side of (5.5.31), picking $\varepsilon \in (0, 1)$ small enough, and using (5.5.13), we can derive estimate (5.5.4). The proof is thus complete. □

The next lemma concerns estimates on u_{xt} and θ_{xt}.

Lemma 5.5.2. *Under assumptions of theorem 5.2.3, the following estimates hold that for any $t > 0$ and $\varepsilon > 0$ small enough,*

$$\|u_{xt}(t)\|^2 + \int_0^t \|u_{xxt}(\tau)\|^2 d\tau \leq C_4 + C_2\varepsilon^2\int_0^t \Big(\|u_{xtt}\|^2 + \|\theta_{xxt}\|^2\Big)(\tau)\, d\tau, \tag{5.5.32}$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(\tau)\|^2 d\tau \leq C_4 + C_2\varepsilon^2\int_0^t \|u_{xxt}(\tau)\|^2 d\tau. \tag{5.5.33}$$

Proof. Similarly to lemma 2.5.2, we can obtain estimate (5.5.32). We derive from (5.2.22) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tx}^2 dx =: \sum_{i=1}^5 D_i(t) \tag{5.5.34}$$

where

$$D_1(t)=-\int_0^L\Big(\frac{r^{2n-2}\kappa\theta_x}{v}\Big)_{tx}\theta_{txx}dx,\ D_2(t)=-\int_0^L\Big(\Big(e_v+P-\delta\frac{(r^{n-1}u)_x}{v})(r^{n-1}u\Big)_x\Big)_{xt}\theta_{tx}dx,$$

$$D_3(t)=2\mu(n-1)\int_0^L(r^{n-2}u^2)_{xt}\theta_{txx}dx,\ D_4(t)=-\int_0^L\Big(e_{\theta tx}\theta_t+\frac{1}{2}e_{\theta t}\theta_{tx}+e_{\theta x}\theta_{tt}\Big)\theta_{tx}dx,$$

$$D_5(t)=\int_0^L(v(S_E)_R)_{tx}\theta_{tx}dx.$$

By virtue of theorems 5.2.1, 5.2.2, lemma 5.5.1 and (5.5.32), and using the embedding theorem and the Young inequality, we derive that for any $\varepsilon\in(0,1)$,

$$D_1(t)\le-(2C_1)^{-1}\|\theta_{txx}\|^2+C_2\Big(\|u_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2\Big),\tag{5.5.35}$$

$$D_2(t)\le\varepsilon^2\|u_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^2}^2+\|\theta_t\|_{H^1}^2+\|u_{xt}\|^2+\|v_x\|_{H^1}^2\Big),\tag{5.5.36}$$

$$D_3(t)\le\varepsilon^2\|\theta_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^1}^2+\|u_t\|^2+\|u_{xt}\|^2\Big),\tag{5.5.37}$$

$$D_4(t)\le\varepsilon^2\|\theta_{txx}\|^2+C_2\varepsilon^{-2}\Big(\|u_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|\theta_x\|_{H^2}^2+\|u_{xt}\|^2+\|v_x\|^2\Big),\tag{5.5.38}$$

$$D_5(t)\le\varepsilon^2\|\theta_{xxt}\|^2+C_1\varepsilon^{-2}\big\|(v(S_E)_R)_t\big\|^2.\tag{5.5.39}$$

Inserting (5.5.35)–(5.5.38) into (5.5.34) and using (5.5.26), we obtain estimate (5.5.33). □

Now we turn to establish some estimates on u, v, θ in $\mathcal{H}^4$.

Lemma 5.5.3. *Under assumptions of theorem 5.2.3, we have, for any* $t>0$,

$$\begin{aligned}&\|u_{tt}\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2+\|\theta_{xt}\|^2\\&\quad+\int_0^t\Big(\|u_{ttx}\|^2+\|u_{xxt}\|^2+\|\theta_{ttx}\|^2+\|\theta_{xxt}\|^2\Big)(\tau)d\tau\le C_4,\end{aligned}\tag{5.5.40}$$

$$\begin{aligned}&\|v_{xxx}\|_{H^1}^2+\|u_{xxx}\|_{H^1}^2+\|\theta_{xxx}\|_{H^1}^2+\|u_{txx}\|^2+\|\theta_{txx}\|^2\\&\quad+\int_0^t\Big(\|u_{tt}\|^2+\|u_{xxt}\|_{H^1}^2+\|\theta_{tt}\|^2+\|\theta_{xxt}\|_{H^1}^2\Big)(\tau)d\tau\le C_4,\end{aligned}\tag{5.5.41}$$

$$\int_0^t\Big(\|v_{xxx}\|_{H^1}^2+\|u_{xxxx}\|_{H^1}^2+\|\theta_{xxxx}\|_{H^1}^2\Big)(\tau)d\tau\le C_4.\tag{5.5.42}$$

Proof. Adding up (5.5.32) and (5.5.33) and choosing ε small enough, we conclude

$$\|u_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t (\|u_{xxt}\|^2 + \|\theta_{xxt}\|^2)(\tau)\,d\tau \le C_4 + C_2\varepsilon^2 \int_0^t \|u_{xtt}(\tau)\|^2\,d\tau. \quad (5.5.43)$$

Now multiplying (5.5.3) and (5.5.4) by ε and ε^2, respectively, then adding up the resultants to (5.5.43) and picking ε small sufficiently, we obtain (5.5.40).

Applying lemma 5.3.9, theorems 5.2.1, 5.2.2 and lemmas 5.5.1, 5.5.2 and using the Hölder inequality, we can derive from equation (5.2.23) that

$$\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_x^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau = \int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} \frac{r^{2-2n}v^2}{\omega^2}S^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau \le C_1, \quad (5.5.44)$$

$$\begin{aligned}
&\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{xx}^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\quad\le C_1\int_0^t\int_0^L (v_x^2+\theta_x^2)\,dx\,d\tau + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_x^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\qquad + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} S^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\quad\le C_1, \qquad (5.5.45)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{xxx}^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\quad\le C_1\int_0^t\int_0^L (v_{xx}^2+\theta_{xx}^2+v_x^2+\theta_x^2)\,dx\,d\tau\\
&\qquad + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{xx}^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\qquad + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_x^2\,d\overrightarrow{\Omega}\,d\epsilon\,dx\,d\tau\\
&\quad\le C_2. \qquad (5.5.46)
\end{aligned}$$

Differentiating (5.2.21) with respect to x and noting that $v_{xxt} = (r^{n-1}u)_{xxx}$, we get

$$\epsilon\frac{\partial}{\partial t}\left(\frac{v_{xx}}{v}\right) - P_v v_{xx} = r^{1-n}u_{tx} + E(x,t) + (1-n)r^{1-2n}vu_t \quad (5.5.47)$$

where

$$E(x,t) = P_{vv}v_x^2 + 2P_{v\theta}\theta_x v_x + 2P_{\theta\theta}\theta_x^2 + P_\theta\theta_{xx} - 2\epsilon\frac{v_x^2}{v^3}(r^{n-1}u)_x + 2\epsilon\frac{v_x}{v^2}(r^{n-1}u)_{xx}.$$

Differentiating (5.5.47) with respect to x, we can get

$$\epsilon\frac{\partial}{\partial t}\Big(\frac{v_{xxx}}{v}\Big)-P_v v_{xxx}=E_1(x,t) \tag{5.5.48}$$

where

$$\begin{aligned}E_1(x,t)&=E_x(x,t)+P_{vx}v_{xx}+\epsilon\Big(\frac{v_{xx}v_x}{v^2}\Big)_t+r^{1-n}u_{txx}+3(1-n)r^{1-2n}vu_{tx}\\&\quad+(1-n)(1-2n)r^{1-3n}v^2u_t+(1-n)r^{1-2n}v_xu_t.\end{aligned}$$

Obviously, from theorem 5.2.2 and lemmas 5.5.1, 5.5.2, it follows that

$$\|E_1(t)\|\le C_2\big(\|v_x\|_{H^1}+\|u_x\|_{H^2}+\|\theta_x\|_{H^2}+\|u_{txx}\|\big) \tag{5.5.49}$$

and

$$\int_0^t\|E_1(\tau)\|^2d\tau\le C_4. \tag{5.5.50}$$

Multiplying (5.5.48) by $\frac{v_{xxx}}{v}$ over $L^2(0,\,L)$, we can obtain

$$\frac{d}{dt}\Big\|\frac{v_{xxx}}{v}\Big\|^2+C_1^{-1}\Big\|\frac{v_{xxx}}{v}\Big\|^2\le C_1\|E_1(t)\|^2 \tag{5.5.51}$$

which, combined with (5.5.50), gives

$$\|v_{xxx}(t)\|^2+\int_0^t\|v_{xxx}(\tau)\|^2d\tau\le C_4. \tag{5.5.52}$$

Thus, noting (5.5.44)–(5.5.46), using the embedding theorem, theorem 5.2.2 and lemmas 5.5.1, 5.5.2, we can derive for any $t>0$,

$$\begin{aligned}&\|u_{xxx}\|^2+\|\theta_{xxx}\|^2+\|u_{xx}\|^2_{L^\infty}+\|\theta_{xx}\|^2_{L^\infty}+\int_0^t\Big(\|u_{xxx}\|^2_{H^1}+\|\theta_{xxx}\|^2_{H^1}\\&\qquad+\|u_{xx}\|^2_{W^{1,\infty}}+\|\theta_{xx}\|^2_{W^{1,\infty}}\Big)(\tau)\,d\tau\le C_4.\end{aligned} \tag{5.5.53}$$

Differentiating (5.2.21), (5.2.22) with respect to t and using (5.5.44)–(5.5.46), theorem 5.2.2 and lemmas 5.5.1, 5.5.2, we can conclude

$$\|u_{txx}\|\le C_1\|u_{tt}\|+C_2\big(\|u_x\|_{H^1}+\|v_x\|+\|\theta_x\|+\|\theta_t\|+\|\theta_{xt}\|+\|u_{xt}\|\big)\le C_4, \tag{5.5.54}$$

$$\|\theta_{txx}\|\le C_1(\|\theta_{tt}\|+\|\mathcal{I}_t\|)+C_2\big(\|u_x\|_{H^1}+\|v_x\|+\|\theta_x\|_{H^2}+\|\theta_t\|_{H^1}+\|u_{xt}\|\big)\le C_4. \tag{5.5.55}$$

Combining with theorems 5.2.1, 5.2.2, (5.5.54), (5.5.55) and lemmas 5.5.1, 5.5.2, we arrive at

$$\|u_{xxxx}(t)\|^2+\|\theta_{xxxx}(t)\|^2+\int_0^t\Big(\|u_{txx}\|^2+\|\theta_{txx}\|^2+\|u_{xxxx}\|^2+\|\theta_{xxxx}\|^2\Big)(\tau)\,d\tau\le C_4. \tag{5.5.56}$$

Therefore,

$$\|u_{xxx}(t)\|_{L^\infty}^2 + \|\theta_{xxx}(t)\|_{L^\infty}^2 + \int_0^t \Big(\|u_{xxx}\|_{L^\infty}^2 + \|\theta_{xxx}\|_{L^\infty}^2\Big)(\tau)\,d\tau \le C_4,\ \forall t > 0. \quad (5.5.57)$$

Now differentiating (5.5.48) with respect to x, we find

$$\epsilon\frac{\partial}{\partial t}\Big(\frac{v_{xxxx}}{v}\Big) - P_v v_{xxxx} = E_2(x,t) \quad (5.5.58)$$

where

$$E_2(x,t) = E_{1x}(x,t) + P_{vx}v_{xxx} + \delta\Big(\frac{v_{xxx}v_x}{v^2}\Big)_t.$$

Using the embedding theorem, lemmas 5.5.1, 5.5.2 and (5.5.52)–(5.5.57), we derive

$$\begin{aligned}
\|E_{xx}(t)\| &\le C_4\big(\|u_x\|_{H^3} + \|\theta_x\|_{H^3} + \|v_x\|_{H^2}\big),\\
\|E_{1x}(t)\| &\le C_1\Bigg(\|E_{xx}(t)\| + \|u_{xxxt}\| + \|u_{xxt}\| + \big\|(P_{vx}v_{xx})_x\big\| + \|v_xu_{xt}\|\\
&\qquad + \|v_xu_t\| + \|v_{xx}u_t\| + \Big\|\Big(\frac{v_xv_{xx}}{v^2}\Big)_{xt}\Big\|\Bigg)\\
&\le C_1\|u_{xxxt}\| + C_4\big(\|u_x\|_{H^3} + \|\theta_x\|_{H^3} + \|v_x\|_{H^2}\big)
\end{aligned}$$

whence

$$\|E_2(t)\| \le C_1\|u_{xxxt}\| + C_4\big(\|u_x\|_{H^3} + \|\theta_x\|_{H^3} + \|v_x\|_{H^2}\big). \quad (5.5.59)$$

It follows from (5.5.11)–(5.5.14) that

$$\int_0^t \Big(\|u_{tt}\|^2 + \|\theta_{tt}\|^2\Big)(\tau)\,d\tau \le C_4. \quad (5.5.60)$$

On the other hand, differentiating (5.2.21) with respect to x and t, and using theorem 5.2.2 and lemma 5.5.1, we can derive

$$\|u_{xxxt}(t)\| \le C_1\|u_{xtt}\| + C_4\big(\|u_x\|_{H^2} + \|\theta_x\|_{H^1} + \|v_x\|_{H^1} + \|\theta_t\|_{H^2}\big). \quad (5.5.61)$$

which, together with (5.5.40) and (5.5.60), gives

$$\int_0^t \|u_{xxxt}(\tau)\|^2\,d\tau \le C_4. \quad (5.5.62)$$

Thus it follows from (5.5.53), (5.5.56), (5.5.59), (5.5.62), theorem 5.2.2 and lemmas 5.5.1, 5.5.2 that

$$\int_0^t \|E_2(\tau)\|^2\,d\tau \le C_4. \quad (5.5.63)$$

Multiplying (5.5.58) by $\frac{v_{xxxx}}{v}$ in $L^2(0,\, L)$, we can get

$$\frac{d}{dt}\left\|\frac{v_{xxxx}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{xxxx}}{v}\right\|^2 \leq C_1\|E_2(t)\|^2 \tag{5.5.64}$$

whence by (5.5.63),

$$\|v_{xxxx}(t)\|^2 + \int_0^t \|v_{xxxx}(\tau)\|^2 d\tau \leq C_4. \tag{5.5.65}$$

Differentiating (5.2.22) with respect to x and t, and using theorem 5.2.2 and lemmas 5.5.1, 5.5.2, we can derive

$$\|\theta_{txxx}\| \leq C_1\|\theta_{ttx}\| + C_2\big(\|u_x\|_{H^3} + \|v_x\|_{H^2} + \|\theta_x\|_{H^3} + \|\theta_{xt}\| + \big\|(v(S_E)_R)_{xt}\big\|\big). \tag{5.5.66}$$

We know that

$$\begin{aligned}\int_0^L (v(S_E)_R)_{xt}^2 dx &\leq C_1\int_0^L ((r^{n-1}u)_{xx}^2(S_E)_R^2 + (r^{n-1}u)_x^2((S_E)_R)_x^2 \\ &\quad + v_x^2((S_E)_R)_t^2 + ((S_E)_R)_{xt}^2)\,dx \\ &=: \sum_{i=1}^4 B_i. \end{aligned}\tag{5.5.67}$$

Using the Hölder inequality and (5.5.44)–(5.5.46), we have

$$B_1 \leq C_1\int_0^L (r^{n-1}u)_{xx}^2\Big(\int_0^{+\infty}\int_{S^{n-1}} S^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)dx \leq C_1\|u_x\|_{H^1}^2, \tag{5.5.68}$$

$$\begin{aligned}B_2 &\leq C_1\int_0^L (r^{n-1}u)_x^2\left(\int_0^{+\infty}\int_{S^{n-1}} S_x^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)dx \\ &\leq C_1\int_0^L ((r^{n-1}u)_x^2 + (r^{n-1}u)_x^2 v_x^2 + (r^{n-1}u)_x^2\theta_x^2)\,dx \\ &\leq C_1\Big(\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|^2\Big),\end{aligned}\tag{5.5.69}$$

$$\begin{aligned}B_3 &\leq C_1\int_0^L v_x^2\Big(\int_0^{+\infty}\int_{S^{n-1}} ((S_E)_R)_t^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)dx \\ &\leq C_1\int_0^L \Big(v_x^2 + (r^{n-1}u)_x^2 v_x^2 + \theta_t^2 v_x^2\Big)dx \\ &\leq C_1\Big(\|u_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|v_x\|^2\Big),\end{aligned}\tag{5.5.70}$$

$$\begin{aligned}B_4 &\le C_1\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}}((S_E)_R)_{xt}^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\Big)dx\\ &\le C_1\Big(\|u_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|v_x\|_{H^1}^2\Big)\\ &\quad + C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{xt}^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx. \end{aligned}\tag{5.5.71}$$

Meanwhile, it follows from (5.2.23) and lemma 5.3.9 that

$$\begin{aligned}&\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_{xt}^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx d\tau\\ &=\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\Big(\frac{r^{1-n}v}{\omega}S\Big)_t^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx d\tau\\ &\le C_1\int_0^t\int_0^L(u_x^2+\theta_t^2)dxd\tau + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^{n-1}} I_t^2 d\overrightarrow{\mathbf{\Omega}}\,d\epsilon dx d\tau\\ &\le C_1. \end{aligned}\tag{5.5.72}$$

Thus we derive from (5.5.66)–(5.5.72) that

$$\int_0^t\|\theta_{txxx}(\tau)\|^2 d\tau\le C_4,\ \forall t>0.\tag{5.5.73}$$

Differentiating (5.2.21) with respect to x three times, using lemmas 5.5.1, 5.5.2 and theorem 5.2.2 and applying the Poincaré inequality, we have

$$\|u_{xxxxx}\|\le C_1\|u_{txxx}\|+C_2\big(\|u_x\|_{H^3}+\|v_x\|_{H^3}+\|\theta_x\|_{H^3}\big).\tag{5.5.74}$$

Thus,

$$\int_0^t\|u_{xxxxx}(\tau)\|^2 d\tau\le C_4,\ \forall t>0.\tag{5.5.75}$$

Using the same technique, we can derive from (5.2.22) that

$$\|\theta_{xxxxx}\|\le C_1(\|\theta_{txxx}\|+\|\mathcal{I}_{xxx}\|)+C_2\big(\|u_x\|_{H^3}+\|v_x\|_{H^3}+\|\theta_x\|_{H^3}\big)\tag{5.5.76}$$

which, together with (5.4.9), (5.5.9), (5.5.10), (5.5.40), (5.5.46), (5.5.52) and (5.5.62), leads to

$$\int_0^t\|\theta_{xxxxx}(\tau)\|^2 d\tau\le C_4,\ \forall t>0.\tag{5.5.77}$$

Hence, we complete the proofs of (5.5.41) and (5.5.42). □

The next lemma concerns estimate on $\|\mathcal{I}(t)\|_{H^5}$.

Lemma 5.5.4. *Under assumptions in theorem 5.2.3, the following estimate holds for any $t>0$,*

$$\|\mathcal{I}(t)\|_{H^5}\le C_4.\tag{5.5.78}$$

Proof. By eqution (5.2.23), we have

$$\begin{aligned}\|\mathcal{I}_{xxxx}(t)\|^2 &= \int_0^L \Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(r^{1-n}vS)_{xxx} d\overrightarrow{\Omega}\, d\epsilon\Big)^2 dx \\ &\leq C_1 \int_0^L \Bigg(\Bigg(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^4+v_x+v_x^2+v_{xx}+v_{xxx})S d\overrightarrow{\Omega}\, d\epsilon\Bigg)^2 \\ &\quad + \Bigg(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^3+v_x+v_{xx})S_x d\overrightarrow{\Omega}\, d\epsilon\Bigg)^2 \\ &\quad + \Bigg(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^2+v_x)S_{xx} d\overrightarrow{\Omega}\, d\epsilon\Bigg)^2 + \Bigg(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}S_{xxx} d\overrightarrow{\Omega}\, d\epsilon\Bigg)^2\Bigg) dx \\ &=: \sum_{i=1}^4 F_i. \end{aligned} \tag{5.5.79}$$

Applying the Holder inequality and the interpolation inequality, we can estimate each term in (5.5.79) as follows,

$$\begin{aligned} F_1 &\leq C_1 \int_0^L (v^8+v_x^2+v_x^4+v_{xx}^2+v_{xxx}^2) \int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2} S^2 d\overrightarrow{\Omega}\, d\epsilon dx \\ &\leq C_1 \int_0^L (v^2+v_x^2+v_x^4+v_{xx}^2+v_{xxx}^2) dx \leq C_1\|v_{xxx}\|^2 \leq C_4, \end{aligned} \tag{5.5.80}$$

$$\begin{aligned} F_2 &\leq C_1 \int_0^L (v^6+v_x^2+v_{xx}^2) \int_0^{\infty}\int_{S^{n-1}} \frac{1}{\omega^2} S_x^2 d\overrightarrow{\Omega}\, d\epsilon dx \\ &\leq C_1 \int_0^L (v^2+v_x^2+v_{xx}^2)(v_x^2+\theta_x^2+\Big(\int_0^{\infty}\int_{S^{n-1}} (\sigma_a+\sigma_s) I_x d\overrightarrow{\Omega}\, d\epsilon)^2\Big) dx \\ &\leq C_1(\|v_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2) \leq C_4, \end{aligned} \tag{5.5.81}$$

$$\begin{aligned} F_3 &\leq C_1 \int_0^L (v^4+v_x^2) \int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2} S_{xx}^2 d\overrightarrow{\Omega}\, d\epsilon dx \\ &\leq C_1 \int_0^L (v^4+v_x^2)(v_{xx}^2+\theta_{xx}^2+v_x^2+\theta_x^2+v_x^2\theta_x^2+\Big(\int_0^{+\infty}\int_{S^{n-1}} (\sigma_a+\sigma_s) I_{xx} d\overrightarrow{\Omega}\, d\epsilon)^2\Big) dx \\ &\leq C_1(\|v_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2) \leq C_4, \end{aligned} \tag{5.5.82}$$

$$\begin{aligned} F_4 &\leq C_1 \int_0^L \int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2} S_{xxx}^2 d\overrightarrow{\Omega}\, d\epsilon dx \\ &\leq C_1\Big(\|v_x\|_{H^2}^2+\|\theta_x\|_{H^2}^2\Big) + C_1\int_0^L \Big(\int_0^{+\infty}\int_{S^{n-1}} (\sigma_a+\sigma_s) I_{xxx} d\overrightarrow{\Omega}\, d\epsilon\Big)^2 dx \\ &\leq C_1\Big(\|v_x\|_{H^2}^2+\|\theta_x\|_{H^2}^2\Big) \leq C_4. \end{aligned} \tag{5.5.83}$$

Inserting (5.5.80)–(5.5.83) into (5.5.79), we have

$$\|\mathcal{I}_{xxxx}(t)\|^2 \le C_4,\ \forall t > 0. \tag{5.5.84}$$

Analogously, we also derive from equation (5.2.23) that

$$\begin{aligned}
\|\mathcal{I}_{xxxxx}(t)\|^2 &= \int_0^L \Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(r^{1-n}vS)_{xxxx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2 dx\\
&\le C_1\int_0^L \Big\{\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^5+v_x+v_x^2+v_{xx}+v_xv_{xx}+v_{xxx}+v_{xxxx})S\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\\
&\quad+\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^4+v_x+v_x^2+v_{xx}+v_{xxx})S_x\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\\
&\quad+\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^3+v_x+v_{xx})S_{xx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\\
&\quad+\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}(v^2+v_x)S_{xxx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2+\Big(\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega}S_{xxxx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\Big\}dx\\
&=: \sum_{i=1}^{5} G_i.
\end{aligned} \tag{5.5.85}$$

Naturally, we can also get the following estimates

$$\begin{aligned}
G_1 &\le C_1\int_0^L \big(v^{10}+v_x^2+v_x^4+v_{xx}^2+v_{xxx}^2+v_x^2v_{xx}^2+v_{xxxx}^2\big)\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2}S^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon dx\\
&\le C_1\|v_{xxxx}\|^2 \le C_4,
\end{aligned} \tag{5.5.86}$$

$$\begin{aligned}
G_2 &\le C_1\int_0^L (v^8+v_x^2+v_x^4+v_{xx}^2+v_{xxx}^2)\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2}S_x^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon dx\\
&\le C_1\int_0^L (v^8+v_x^2+v_x^4+v_{xx}^2+v_{xxx}^2)\Big(v_x^2+\theta_x^2+\Big(\int_0^{+\infty}\int_{S^{n-1}}(\sigma_a+\sigma_s)I_x\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\Big)dx\\
&\le C_1\Big(\|v_x\|_{H^3}^2+\|\theta_x\|^2\Big) \le C_4,
\end{aligned} \tag{5.5.87}$$

$$\begin{aligned}
G_3 &\le C_1\int_0^L (v^6+v_x^2+v_{xx}^2)\int_0^{+\infty}\int_{S^{n-1}} \frac{1}{\omega^2}S_{xx}^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon dx\\
&\le C_1\int_0^L (v^6+v_x^2+v_{xx}^2)\Big(v_{xx}^2+\theta_{xx}^2+v_x^2+\theta_x^2+v_x^2\theta_x^2+\Big(\int_0^{+\infty}\int_{S^{n-1}}(\sigma_a+\sigma_s)I_{xx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\Big)^2\Big)dx\\
&\le C_1\Big(\|v_x\|_{H^2}^2+\|\theta_x\|_{H^1}^2\Big) \le C_4,
\end{aligned} \tag{5.5.88}$$

$$\begin{aligned} G_4 &\le C_1\int_0^L (v^4+v_x^2)\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{\omega^2}S_{xxx}^2 d\overrightarrow{\Omega}\,d\epsilon dx\\ &\le C_1(\|v_{xx}\|_{H^1}^2+\|\theta_{xx}\|_{H^1}^2)+C_1\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}}(\sigma_a+\sigma_s)I_{xxx}d\overrightarrow{\Omega}\,d\epsilon\Big)^2dx\\ &\le C_1\Big(\|v_{xx}\|_{H^1}^2+\|\theta_{xx}\|_{H^1}^2\Big)\le C_4, \end{aligned} \tag{5.5.89}$$

$$\begin{aligned} G_5 &\le C_1\int_0^L\int_0^{+\infty}\int_{S^{n-1}}\frac{1}{\omega^2}S_{xxxx}^2 d\overrightarrow{\Omega}\,d\epsilon dx\\ &\le C_1\Big(\|v_{xx}\|_{H^2}^2+\|\theta_{xx}\|_{H^2}^2\Big)+C_1\int_0^L\Big(\int_0^{+\infty}\int_{S^{n-1}}(\sigma_a+\sigma_s)I_{xxxx}d\overrightarrow{\Omega}\,d\epsilon\Big)^2dx\\ &\le C_1\Big(\|v_{xx}\|_{H^2}^2+\|\theta_{xx}\|_{H^2}^2\Big)\le C_4. \end{aligned} \tag{5.5.90}$$

Inserting (5.5.86)–(5.5.90) into (5.5.85), we get

$$\|\mathcal{I}_{xxxxx}(t)\|^2\le C_4,\ \forall t>0, \tag{5.5.91}$$

which, along with (5.5.84), leads to (5.5.78). □

By lemmas 5.5.1–5.5.4, we have proved the global existence of solutions in $\mathcal{H}^4$ to the problem (5.2.20)–(5.2.23) in $\mathcal{H}^4$ with arbitrary initial datum $(u_0, v_0, \theta_0, \mathcal{I}_0)\in\mathcal{H}^4$ and the uniqueness of solutions follows from that of solutions in $\mathcal{H}^1$ or in $\mathcal{H}^2$.

5.5.2 Asymptotic Behavior in $\mathcal{H}^4$

In this subsection, we shall show the asymptotic behavior of solutions.

Lemma 5.5.5. *Under assumptions in theorem 5.2.3, we have, as $t\to+\infty$,*

$$\|v(t)-\overline{v}\|_{H^4}\to 0,\ \|u(t)\|_{H^4}\to 0,\ \big\|\theta(t)-\overline{\theta}\big\|_{H^4}\to 0. \tag{5.5.92}$$

Proof. Differentiating (5.2.20) with respect to x three times and then multiplying the resultant by v_{xxx} over $L^2(0,\,L)$, we have

$$\frac{d}{dt}\|v_{xxx}\|^2\le C_2\Big(1+\|u_{xxx}\|^2+\|u_{xxxx}\|^2+\|v_{xxx}\|^2\Big) \tag{5.5.93}$$

which, along with (5.5.56) and theorems 5.2.1, 5.2.2 and 1.2.10, leads to, as $t\to+\infty$,

$$\|v_{xxx}(t)\|\to 0. \tag{5.5.94}$$

Similarly, differentiating (5.2.20) with respect to x four times, we can obtain

$$\frac{d}{dt}\|v_{xxxx}\|^2 \le C_2\Big(1+\|u_{xxx}\|^2+\|u_{xxxx}\|^2+\|u_{xxxxx}\|^2+\|v_{xxxx}\|^2\Big) \tag{5.5.95}$$

which, along with (5.5.75) and theorems 5.2.1, 5.2.2 and 1.2.10, yields, as $t \to +\infty$,

$$\|v_{xxxx}(t)\| \to 0, \tag{5.5.96}$$

which, combined with the Poincaré inequality, gives, as $t \to +\infty$,

$$\|v(t)-\overline{v}\|_{H^4} \to 0. \tag{5.5.97}$$

From lemma 5.5.2, we can easily get

$$\begin{aligned}\frac{d}{dt}\|\theta_{xt}(t)\|^2+\|\theta_{xxt}(t)\|^2 \le \varepsilon\Big(\|\theta_{xxxt}(t)\|^2+\|v_{xxt}(t)\|^2\Big)+C_2\Big(\|v_{xxx}(t)\|^2+\|v_{xt}(t)\|^2\\ +\|\theta_{xxx}(t)\|^2+\|\theta_{xt}(t)\|^2+\|u_x(t)\|^2+\|\mathcal{I}_x(t)\|^2\Big),\end{aligned} \tag{5.5.98}$$

which, combined with theorem 1.2.10, leads to, as $t \to +\infty$,

$$\|\theta_{xt}(t)\| \to 0. \tag{5.5.99}$$

Noting that, as $t \to +\infty$,

$$\|\mathcal{I}(t)\|_{H^1} \to 0 \tag{5.5.100}$$

and using (5.4.11), we have, as $t \to +\infty$,

$$\|\theta_{xxx}(t)\| \to 0. \tag{5.5.101}$$

From lemma 5.5.1, we can easily obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{d}{dt}\|\theta_{tt}(t)\|^2+\|\theta_{xxt}(t)\|^2 \le \varepsilon\Big(\|\theta_{xtt}(t)\|^2+\|v_{xtt}(t)\|^2\Big)+C_2\Big(\|v_{xx}(t)\|^2+\|v_{xt}(t)\|^2\\ +\|\theta_{tt}(t)\|^2+\|\theta_{xt}(t)\|^2+\|\mathcal{I}_{xx}(t)\|^2\Big),\end{aligned} \tag{5.5.102}$$

which, by theorem 1.2.10, leads to, as $t \to +\infty$,

$$\|\theta_{tt}(t)\| \to 0. \tag{5.5.103}$$

On the other hand, we have

$$\|\theta_{xxt}(t)\| \le C_2(\|\theta_{tt}(t)\|+\|v_x(t)\|+\|u_x(t)\|+\|\theta_t(t)\|+\|\theta_{xt}(t)\|+\|\theta_{xxx}(t)\|+\|\mathcal{I}_{xx}(t)\|),$$

which, combined with theorem 1.2.10 and (5.5.98)–(5.5.103), yields as $t \to +\infty$,

$$\|\theta_{xxt}(t)\| \to 0. \tag{5.5.104}$$

Thus it follows from (5.5.104) and (5.5.10) that, as $t \to +\infty$,

$$\|\theta_{xxxx}(t)\| \to 0, \tag{5.5.105}$$

which, obviously, along with theorem 1.2.10 and the Poincaré inequality, leads to our desired estimate $\|\theta(t)-\overline{\theta}\|_{H^4}\rightarrow 0$, as $t\rightarrow+\infty$.

Analogously, applying lemmas 5.5.1, 5.5.2 and 1.2.10, we can derive from (5.2.21) that $\|u(t)\|_{H^4}\rightarrow 0$. Here we omit it. □

Lemma 5.5.6. *Under assumptions in theorem 5.2.3, we have, as $t\rightarrow+\infty$,*

$$\|\mathcal{I}(t)\|_{H^5}\rightarrow 0. \tag{5.5.106}$$

Proof. We can derive from the derivations of (5.5.80)–(5.5.83) that

$$F_1\leq C_1\|v_{xxx}\|^2, \tag{5.5.107}$$

$$F_2\leq C_1\left(\|v_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2\right), \tag{5.5.108}$$

$$F_3\leq C_1\left(\|v_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\mathcal{I}_{xx}\|^2\right), \tag{5.5.109}$$

$$F_4\leq C_1\left(\|v_x\|_{H^2}^2+\|\theta_x\|_{H^2}^2+\|\mathcal{I}_x\|^2+\|\mathcal{I}_{xx}\|^2\right). \tag{5.5.110}$$

Applying (5.5.79), (5.5.107)–(5.5.110) and theorems 5.2.1, 5.2.2, we have, as $t\rightarrow+\infty$,

$$\|\mathcal{I}_{xxxx}(t)\|^2\rightarrow 0. \tag{5.5.111}$$

Similarly, combining (5.5.85)–(5.5.90), (5.5.111) and using theorems 5.2.1, 5.2.2, we also have, as $t\rightarrow+\infty$,

$$\|\mathcal{I}_{xxxxx}(t)\|^2\rightarrow 0. \tag{5.5.112}$$

Therefore, we complete the proof. □

Till now we have completed the proof of theorem 5.2.3. □

5.6 Bibliographic Comments

Note that there are some known results on the system (5.1.2)–(5.1.4). First, Lowrie, *et al.* [85], Buet and Després [8] have investigated the asymptotic regimes in the inviscid case and Dubroca and Feugeas [24], Lin [83] and Lin *et al.* [84] considered the numerical aspects. As regards the existence of solutions in the multi-dimensional case, a proof of local-in-time existence and blow-up of solutions has been proposed by Zhong and Jiang [145] and a simplified version of the system has been investigated by Golse and Perthame [46]. Li and Zhu [82] have investigated the blow-up of smooth solutions under some sufficient conditions for the Cauchy problem.

For the one-dimensional initial-boundary value problem, Ducomet and Nečasová [31] have considered global existence and uniqueness of weak solutions in $H^1 \times H^1 \times H^1 \times H^1$ and Qin *et al.* [106] also proved the global existence and large-time behavior of solutions in $H^i \times H_0^i \times H^i \times H^{i+1}$ $(i = 1, 2)$ for the infrarelativistic model. Ducomet and Nečasová [32] obtained the asymptotic behavior of global strong solutions. In the pure scattering case, Ducomet and Nečasová [33] investigated the asymptotic behavior of a motion of a viscous heat-conducting one-dimensional gas with radiation and Qin *et al.* [107] also proved the large-time behavior of solutions in $H^i \times H_0^i \times H^i \times H^{i-1}$ $(i = 2, 3)$.

Therefore, we study global solvability for this model with special symmetric properties in this chapter and the next chapter. For spherically symmetric Navier–Stokes equations, we would like to refer to [48, 51, 64, 67, 139] and the references therein.

Chapter 6

Global Existence and Asymptotic Behavior of Cylindrically Symmetric Solutions for the 3D Infrarelativistic Model with Radiation

6.1 Introduction

In this chapter, we shall consider the motion of the compressible 3D viscous gas with radiation. The model is the same as that of chapter 5. However, here we only consider the 3D space case. The role of the radiation is primarily one of transporting energy by radiative process and thus the radiation field affects the dynamics of the field. As we know, the aim of radiation hydrodynamics is to incorporate the effects of radiation in the conventional hydrodynamics framework. The motion of the fluid is described by standard fluid mechanics giving the evolution of the mass density $\rho = \rho(x, t)$, the velocity field $\mathbf{U} = \mathbf{U}(x, t)$ and the absolute temperature $\theta = \theta(x, t)$, for all $t > 0$ and all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The dynamics of radiation is described by a transport equation for the specific radiative intensity $I = I(x, t, \epsilon, \overrightarrow{\Omega})$ which depends on the frequency $\epsilon \geq 0$ and the direction vector $\overrightarrow{\Omega} \in S^2$, where $\overrightarrow{\Omega}$ represents the direction of travel of the photon (it requires two angular variables to specify $\overrightarrow{\Omega}$) and S^2 denotes the unit sphere in $\mathbb{R}^3$. Moreover this equation in radiative transfer is driven by the so called radiative transfer integro-differential equation introduced and discussed by Chandrasekhar [10]. Under the consideration of the three basic interactions between photons and matter, namely, absorption, scattering and emission, suppose that the matter is in local thermodynamical equilibrium and radiation is present with coupling terms between matter and radiation, the coupled system reads (see, *e.g.*, [88, 96, 97])

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$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{U}) = 0, & (6.1.1)\\ \partial_t(\rho\mathbf{U}) + \operatorname{div}(\rho\mathbf{U}\otimes\mathbf{U}) = -\operatorname{div}\overrightarrow{\mathbf{\Pi}} - \overrightarrow{S}_F, & (6.1.2)\\ \partial_t(\rho e + \operatorname{div}(\rho e\mathbf{U}) = -\operatorname{div}Q - \overrightarrow{D} : \overrightarrow{\mathbf{\Pi}} - S_E, & (6.1.3)\\ \dfrac{1}{c}\dfrac{\partial I(\epsilon,\overrightarrow{\mathbf{\Omega}})}{\partial t} + \overrightarrow{\mathbf{\Omega}}\cdot\nabla I(\epsilon,\overrightarrow{\mathbf{\Omega}}) = S_t(\epsilon,\overrightarrow{\mathbf{\Omega}}), & (6.1.4)\end{cases}$$

where $e = e(x, t)$, $Q = Q(x, t)$ are the internal energy and the heat flux, respectively, and c denotes the velocity of light, $\overrightarrow{\mathbf{\Pi}} = -P(\rho,\theta)\overrightarrow{I} + \overrightarrow{\pi}$ represents the material stress tensor for a Newtonian fluid with the viscous contribution $\overrightarrow{\pi} = 2\mu\overrightarrow{D} + \lambda\operatorname{div}\mathbf{U}\overrightarrow{I}$ with $\mu > 0$ and $3\lambda + 2\mu \geq 0$, and the strain tensor $\overrightarrow{D}$ such that $\overrightarrow{D}_{ij} = \frac{1}{2}\left(\frac{\partial\mathbf{U}_i}{\partial x_j} + \frac{\partial\mathbf{U}_j}{\partial x_i}\right)$, $P(\rho, \theta)$ is the pressure and the coupling terms are

$$\begin{aligned} S_t(\epsilon,\overrightarrow{\mathbf{\Omega}}) \equiv S_t(x,t,\epsilon,\overrightarrow{\mathbf{\Omega}}) &= \sigma_a(\epsilon,\overrightarrow{\mathbf{\Omega}})(B(\epsilon,\theta) - I(\epsilon,\overrightarrow{\mathbf{\Omega}}))\\ &\quad + \int_0^{+\infty} d\epsilon' \int_{S^2} \sigma_s(\epsilon'\to\epsilon,\overrightarrow{\mathbf{\Omega}}'\cdot\overrightarrow{\mathbf{\Omega}})\Big(\frac{\epsilon}{\epsilon'}I(\epsilon',\overrightarrow{\mathbf{\Omega}}'\Big)\\ &\quad - \sigma_s(\epsilon\to\epsilon',\overrightarrow{\mathbf{\Omega}}\cdot\overrightarrow{\mathbf{\Omega}}')I(\epsilon,\overrightarrow{\mathbf{\Omega}})I(\epsilon',\overrightarrow{\mathbf{\Omega}}'))\,d\overrightarrow{\mathbf{\Omega}}', \end{aligned}$$

the radiative energy source

$$S_E(x,t) := \int_0^{+\infty} d\epsilon \int_{S^2} S_t(\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$

the radiative flux

$$\overrightarrow{S}_F(x,t) := \frac{1}{c}\int_0^{+\infty} d\epsilon \int_{S^2} \overrightarrow{\mathbf{\Omega}}S_t(\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}},$$

and Planck's function $B(\epsilon, \theta)$ describes the frequency-temperature black body distribution. The thermo-radiative flux Q satisfies the Fourier's law

$$Q = Q(\rho,\theta) = -\kappa(\rho,\theta)\nabla\theta, \tag{6.1.5}$$

where $\kappa(\rho, \theta)$ is the heat conductivity coefficient.

Now we assume $G \subset \mathbb{R}^3$ is a bounded domain and consider the following boundary conditions

$$\mathbf{U}|_{\partial G} = 0, \quad Q|_{\partial G} = 0, \quad I|_{\partial G} = 0, \tag{6.1.6}$$

together with the initial conditions

$$(\rho,\mathbf{U},\theta,I)(x,0) = (\rho_0,\mathbf{U}_0,\theta_0,I_0)(x), \qquad x \in G. \tag{6.1.7}$$

In what follows, letters C_i $(i = 1, 2, 4)$ will denote the universal positive constants depending on the norms of initial datum $(u_0, v_0, \theta_0, \mathcal{I}_0)$ in $\mathcal{H}^i$ (see below the definition of $\mathcal{H}^i$) $(i = 1, 2, 4)$ but being independent of t, respectively.

In this chapter, we study the cylindrically symmetric solutions of the problem (6.1.1)–(6.1.7). The strategy we use to prove our desired results consists in an adaptation to the radiative case of the ideas of Qin [99, 103], Ducomet and Nečasová [31–33] and Qin *et al.* [106, 107] and Umehara and Tani [134, 135]. Motivated by Cui and Yao [18], we remove the smallness of initial data and establish uniform point-wise positive upper and lower bounds of the specific volume by means of the methods of [66, 79]. These results in this chapter are selected from [118].

6.2 Reformation and Main Results

Now we assume that the fluid motion is small enough with respect to the velocity of light c so that we can drop all the $\frac{1}{c}$ factors in the previous formulation and then get an infrarelativistic model of a compressible Navier–Stokes system coupled to the radiative transfer equation.

6.2.1 Reformation of Model

Now we restrict ourselves to the flows between two circular coaxial cylinders and the corresponding solutions depend only on the radial variable $r = \sqrt{x_1^2 + x_2^2}$, $G = \{r\colon 0 < a < r < b < +\infty\}$ and the time variable t. Then, the three-dimensional infrareletivistic model with radiation reduces to the following system of the form

$$\begin{cases}
\rho_t + (\rho u)_r + \dfrac{\rho u}{r} = 0, & (6.2.1)\\
\rho\left(u_t + uu_r - \dfrac{v^2}{r}\right) + P_r - (2\mu + \lambda)\left(u_r + \dfrac{u}{r}\right)_r = 0, & (6.2.2)\\
\rho\left(v_t + uv_r + \dfrac{uv}{r}\right) - \mu\left(v_r + \dfrac{v}{r}\right)_r = 0, & (6.2.3)\\
\rho(w_t + uw_r) - \mu\left(w_{rr} + \dfrac{w_r}{r}\right) = 0, & (6.2.4)\\
\rho e_t - (\kappa\theta_r)_r - \dfrac{\kappa\theta_r}{r} + P\left(u_r + \dfrac{u}{r}\right) - \mathcal{Q} = -(S_E)_R, & (6.2.5)\\
\omega(rI)_r = rS, & (6.2.6)
\end{cases}$$

where

$$\mathcal{Q} = \lambda\left(u_r + \frac{u}{r}\right)^2 + \mu\left(\left(v_r - \frac{v}{r}\right)^2 + w_r^2 + 2u_r^2 + 2\left(\frac{u}{r}\right)^2\right)$$

and is supplemented with the initial and boundary conditions

$$(u, v, w)|_{r=a,b} = 0, \quad \theta_r|_{r=a,b} = 0, \quad I|_{r=a,b} = 0, \quad t \geq 0, \tag{6.2.7}$$

$$(\rho, u, v, w, \theta, I)(r, 0) = (\rho_0, u_0, v_0, w_0, \theta_0, I_0)(r), \quad r \in (a, b). \tag{6.2.8}$$

The velocity vector $\overrightarrow{\mathbf{v}} = (u, v, w)$ is given by the radial, angular and axial velocities, respectively, ω denotes the cosine of the angular between the position vector x and the transport vector $\overrightarrow{\mathbf{\Omega}}$.

We are interested in the global existence and asymptotic behavior of solutions to the initial boundary value problem for (6.2.1)–(6.2.8). Similarly to chapter 2, we use $\tau := \frac{1}{\rho}$ to represent the specific volume and transform equations (6.2.1)–(6.2.6) in the Eulerian coordinates into the Lagrangian coordinates as follows,

$$\begin{cases} \tau_t = (ru)_x, \quad 0<x<L,\ t>0, & (6.2.9)\\ u_t = r\left(\delta\dfrac{(ru)_x}{\tau} - P\right)_x + \dfrac{v^2}{r}, & (6.2.10)\\ v_t = \mu r\left(\dfrac{(rv)_x}{\tau}\right)_x - \dfrac{uv}{r}, & (6.2.11)\\ w_t = \mu r\left(\dfrac{(rw)_x}{\tau}\right)_x + \dfrac{\tau w}{r^2}, & (6.2.12)\\ \left(e + \dfrac{1}{2}|\overrightarrow{\mathbf{v}}|^2\right)_t = \left(\kappa\dfrac{r^2\theta_x}{\tau} + \left(\delta\dfrac{(ru)_x}{\tau} - P\right)ru + \mu\left(\dfrac{rv(rv)_x}{\tau} + \dfrac{rw(rw)_x}{\tau}\right)\right)_x & \\ \qquad\qquad - 2\mu\left(u^2 + v^2 + \dfrac{1}{2}w^2\right)_x - \tau(S_E)_R, & (6.2.13)\\ \omega(rI)_x = \tau S, & (6.2.14) \end{cases}$$

with $\delta = \lambda + 2\mu$ and the initial boundary conditions (6.2.7) and (6.2.8) become

$$\overrightarrow{\mathbf{v}}(x,t)|_{x=0,L} = 0, \quad \theta_x(x,t)|_{x=0,L} = 0, \quad t \geq 0, \tag{6.2.15}$$

$$I(0,t) = 0 \ \text{ for } \ \omega \in (0,1), \quad I(L,t) = 0 \ \text{ for } \ \omega \in (-1,0), \quad t \geq 0, \tag{6.2.16}$$

$$\tau(x,0) = \tau_0(x), \quad \overrightarrow{\mathbf{v}}(x,0) = \overrightarrow{\mathbf{v}}_0(x), \quad \theta(x,0) = \theta_0(x), \quad I(x,0) = I_0(x), \quad x \in \mathbf{\Omega}. \tag{6.2.17}$$

It is well known that $r(x, t)$ is determined by

$$r(x,t) = r_0(x) + \int_0^t u(x,s)\,ds, \quad r_0(x) := \left(a^2 + 2\int_0^x \tau_0(y)\,dy\right)^{\frac{1}{2}},$$

that is,

$$r_t = u, \quad rr_x = \tau, \quad r|_{x=0} = a, \quad r|_{x=L} = b. \tag{6.2.18}$$

Thus, we can derive by the same arguments as those in [65, 99, 109] that for all $(x, t) \in [0, L] \times [0, +\infty)$,

$$0 < a = r(0,t) \leq r(x,t) \leq r(L,t) = b. \tag{6.2.19}$$

In this symmetric model, the radiative energy source

$$(S_E)_R \equiv (S_E)_R(x, t) = \int_0^{+\infty} d\epsilon \int_{S^2} S(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}})\, d\overrightarrow{\mathbf{\Omega}}$$

and source term S is defined as

$$S \equiv S(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) = S_{a,e}(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) + S_s(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}),$$

where the absorption–emission term is $S_{a,e}(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) = \sigma_a(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}})(B(x, t; \epsilon) - I(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}))$ and the scattering term is $S_s(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}) = \sigma_s(x, t; \epsilon)(\widetilde{I}(x, t; \epsilon) - I(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}}))$ where $B(\theta;\, \epsilon) = B(x,\, t;\, \epsilon)$ is a function of temperature and frequency describing the equilibrium state and

$$\widetilde{I}(x, t; \epsilon) = \frac{1}{4\pi} \int_{S^2} I(x, t; \epsilon, \overrightarrow{\mathbf{\Omega}})\, d\overrightarrow{\mathbf{\Omega}}.$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_\tau(\tau, \theta) = -P(\tau, \theta) + \theta P_\theta(\tau, \theta).$$

We assume that e, P, σ_a, σ_s, κ and B are C^{i+1} $(i = 1,\ 2,\ 4)$ functions on $0 < \tau < +\infty$ and $0 \leq \theta < +\infty$ and satisfy the following growth conditions:

(A_1) $e(\tau, 0) \geq 0, \quad c_1^{-1}(1 + \theta^d) \leq e_\theta(\tau, \theta) \leq c_1(1 + \theta^d),$

(A_2) $-\, c_2^{-1}\tau^{-2}(1 + \theta^{1+d}) \leq P_\tau(\tau, \theta) \leq -\, c_2\tau^{-2}(1 + \theta^{1+d}),$

(A_3) $|P_\theta(\tau, \theta)| \leq c_3\tau^{-1}(1 + \theta^d),$

(A_4) $c_4^{-1}(1 + \theta^{1+d}) \leq \tau P(\tau, \theta) \leq c_4(1 + \theta^{1+d}), \quad P_\tau(\tau, \theta_0) \leq 0,$

(A_5) $0 \leq P(\tau, \theta) \leq c_5(1 + \theta^{1+d}),$

(A_6) $c_6^{-1}(1 + \theta^q) \leq \kappa(\tau, \theta) \leq c_6(1 + \theta^q),$

(A_7) $|\kappa_\tau(\tau, \theta)| + |\kappa_{\tau\tau}(\tau, \theta)| \leq c_7(1 + \theta^q),$

(A_8) $\tau\sigma_a(\tau, \theta; \epsilon, \overrightarrow{\mathbf{\Omega}})B^m(\theta; \epsilon) \leq c_8|\omega|\theta^{1+\alpha}f(\epsilon, \overrightarrow{\mathbf{\Omega}}), \quad \text{for } m = 1, 2,$

(A_9) $0 < \sigma_a(\tau, \theta; \epsilon, \overrightarrow{\mathbf{\Omega}}) \leq c_9|\omega|^2 g(\epsilon, \overrightarrow{\mathbf{\Omega}}),$

(A_{10}) $(\sigma_a + |(\sigma_a)_\tau| + |(\sigma_a)_\theta|)(\tau, \theta; \epsilon, \overrightarrow{\mathbf{\Omega}})(1 + B(\theta; \epsilon) + |B_\theta(\theta; \epsilon)| + |B_{\theta\theta}(\theta; \epsilon)|) \leq c_{10}|\omega|h(\epsilon, \overrightarrow{\mathbf{\Omega}}),$

(A_{11}) $0 < \sigma_s(\tau, \theta; \epsilon) \leq c_{11}|\omega|^2 k(\epsilon, \overrightarrow{\mathbf{\Omega}}),$

(A_{12}) $(|(\sigma_a)_{\tau\tau}| + |(\sigma_a)_{\tau\theta}| + |(\sigma_a)_{\theta\theta} + |(\sigma_a)_{\tau\tau\tau}| + |(\sigma_a)_{\tau\tau\theta}| + |(\sigma_a)_{\tau\theta\theta}| + |(\sigma_a)_{\theta\theta\theta}| + |(\sigma_a)_{\theta\theta\theta\theta}|$
$+ |(\sigma_a)_{\tau\theta\theta\theta}| + |(\sigma_a)_{\tau\tau\theta\theta}| + |(\sigma_a)_{\tau\tau\tau\theta}| + |(\sigma_a)_{\tau\tau\tau\tau}|)(\tau, \theta; \epsilon, \overrightarrow{\mathbf{\Omega}})(1 + B(\theta; \epsilon) + |B_\theta(\theta; \epsilon)|$
$+ |B_{\theta\theta}(\theta; \epsilon)| + |B_{\theta\theta\theta}(\theta; \epsilon)| + |B_{\theta\theta\theta\theta}(\theta; \epsilon)|) \leq c_{12}|\omega|l(\epsilon, \overrightarrow{\mathbf{\Omega}}),$

(A_{13}) $(|(\sigma_s)_\tau| + |(\sigma_s)_\theta| + |(\sigma_s)_{\tau\tau}| + |(\sigma_s)_{\tau\theta}| + |(\sigma_s)_{\theta\theta}| + |(\sigma_s)_{\tau\tau\tau}| + |(\sigma_s)_{\tau\tau\theta}| + |(\sigma_s)_{\tau\theta\theta}|$
$+ |(\sigma_s)_{\theta\theta\theta}| + |(\sigma_s)_{\theta\theta\theta\theta}| + |(\sigma_s)_{\tau\theta\theta\theta}| + |(\sigma_s)_{\tau\tau\theta\theta}| + |(\sigma_s)_{\tau\tau\tau\theta}| + |(\sigma_s)_{\tau\tau\tau\tau}|)(\tau, \theta; \epsilon)$
$\leq c_{13}|\omega|\mathcal{M}(\epsilon, \overrightarrow{\mathbf{\Omega}})$

where the numbers c_j $(j = 1,\ldots, 13)$ are positive constants, $d \in [0,\ 1]$, $q \geq 1 + d$, $0 \leq \alpha \leq 1 + d$ and the nonnegative functions $f, g, h, k, l, \mathcal{M}$ are such that for any $\gamma > 0$,

$$f, g, h, k, l, \mathcal{M} \in L^{1+\gamma}(\mathbb{R}^+ \times S^2) \bigcap L^\infty(\mathbb{R}^+ \times S^2).$$

We assume that the viscosity coefficient μ is a positive constant. In the following, we denote

$$\mathcal{I}(x,t) := \int_0^{+\infty} d\epsilon \int_{S^2} I(x,t;\epsilon,\overrightarrow{\mathbf{\Omega}})\, d\overrightarrow{\mathbf{\Omega}} \tag{6.2.20}$$

for the integrated radiative intensity. In particular, $\mathcal{I}(x,0) = \mathcal{I}_0 = \int_0^{+\infty} d\epsilon \int_{S^2} I(x,0;\epsilon,\overrightarrow{\mathbf{\Omega}})\, d\overrightarrow{\mathbf{\Omega}}$.

6.2.2 Main Theorems

We define some classes of functions as

$$\begin{aligned}\mathcal{H}^1 = \{(\tau,\overrightarrow{\mathbf{v}},\theta,\mathcal{I}) \in H^1[0,L] \times (H_0^1[0,L])^3 \times H^1[0,L] \times H^2[0,L] : \tau(x) > 0,\ \theta(x) > 0,\\ \overrightarrow{\mathbf{v}}(0) = \overrightarrow{\mathbf{v}}(L) = 0,\ \mathcal{I}(0) = 0 \text{ for } \omega \in (0,1) \text{ and } \mathcal{I}(L) = 0 \text{ for } \omega \in (-1,0)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^i = \{(\tau,\overrightarrow{\mathbf{v}},\theta,\mathcal{I}) \in H^i[0,L] \times (H^i[0,L])^3 \times H^i[0,L] \times H^{i+1}[0,L] : \tau(x) > 0,\ \theta(x) > 0,\\ \overrightarrow{\mathbf{v}}(0) = \overrightarrow{\mathbf{v}}(L) = 0,\ \theta'(0) = \theta'(L) = 0,\ \mathcal{I}(0) = 0 \text{ for } \omega \in (0,1)\\ \text{and } \mathcal{I}(L) = 0 \text{ for } \omega \in (-1,0)\},\quad i = 2,4,\end{aligned}$$

which become the metric spaces when equipped with the metrics induced from the usual norms.

We are now in a position to state our main theorems.

Theorem 6.2.1. *Assume that the initial data* $(\tau_0,\overrightarrow{\mathbf{v}}_0,\theta_0,\mathcal{I}_0) \in \mathcal{H}^1$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\tau(t),\overrightarrow{\mathbf{v}}(t),\theta(t),\mathcal{I}(t)) \in L^\infty([0,+\infty),\mathcal{H}^1)$ *to problem (6.2.9)–(6.2.17) verifying for all* $(x,\ t) \in [0,\ L] \times [0,\ +\infty)$,

$$0 < C_1^{-1} \leq \tau(x,t) \leq C_1,\quad 0 < C_1^{-1} \leq \theta(x,t) \leq C_1$$

and

$$\begin{aligned}&\|\tau(t) - \overline{\tau}\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}(t)\right\|_{H^1}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^1}^2 + \|\mathcal{I}(t)\|_{H^2}^2 + \|r(t) - \overline{r}\|_{H^2}^2 + \|r_t(t)\|_{H^1}^2\\ &\quad + \int_0^t \left(\|\tau - \overline{\tau}\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}\right\|_{H^2}^2 + \left\|\theta - \overline{\theta}\right\|_{H^2}^2 + \|r - \overline{r}\|_{H^2}^2 + \|r_t\|_{H^2}^2 + \|\theta_t\|^2\right)(s)\,ds\\ &\quad + \int_0^t \int_0^L \int_0^{+\infty} \int_{S^2} I_t^2\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\, dx\, ds \leq C_1.\end{aligned}$$

Moreover, we have, as $t \to +\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^1}\to 0,\quad \|\overrightarrow{\mathbf{v}}(t)\|_{H^1}\to 0,\quad \|\theta(t)-\overline{\theta}\|_{H^1}\to 0,\quad \|\mathcal{I}(t)\|_{H^2}\to 0$$

where $\overline{\tau}=\overline{\tau_0}=L^{-1}\int_0^L \tau_0(x)dx$, *the constant* $\overline{\theta}>0$ *is determined by* $e(\overline{\tau},\overline{\theta}) = \int_0^L\left(\frac{1}{2}|\overrightarrow{\mathbf{v}}_0|^2+e(\tau_0,\theta_0)\right)dx$ *and* $\overline{r}=(a^2+2\overline{\tau}x)^{\frac{1}{2}}$.

Theorem 6.2.2. *Assume that the initial data* $(\tau_0,\overrightarrow{\mathbf{v}}_0,\theta_0,\mathcal{I}_0)\in\mathcal{H}^2$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\tau(t),\overrightarrow{\mathbf{v}}(t),\theta(t),\mathcal{I}(t))\in L^\infty([0,+\infty),\mathcal{H}^2)$ *to problem (6.2.9)–(6.2.17) satisfying for all* $(x,t)\in[0,L]\times[0,+\infty)$,

$$\begin{aligned}&\|\tau(t)-\overline{\tau}\|_{H^2}^2+\|\overrightarrow{\mathbf{v}}(t)\|_{H^2}^2+\|\theta(t)-\overline{\theta}\|_{H^2}^2+\|\mathcal{I}(t)\|_{H^3}^2+\|r(t)-\overline{r}\|_{H^3}^2+\|r_t(t)\|_{H^2}^2+\|\overrightarrow{\mathbf{v}}_t(t)\|^2\\&\quad+\|\theta_t(t)\|^2+\int_0^t\Big(\|\tau-\overline{\tau}\|_{H^2}^2+\|\overrightarrow{\mathbf{v}}\|_{H^3}^2+\|\theta-\overline{\theta}\|_{H^3}^2+\|r-\overline{r}\|_{H^3}^2+\|r_t\|_{H^3}^2\\&\quad+\|\overrightarrow{\mathbf{v}}_{xt}\|^2+\|\theta_{xt}\|^2\Big)(s)\,ds\le C_2\end{aligned}$$

where $\overline{\tau},\overline{\theta}>0$ *and* $\overline{r}$ *are also the same as those of theorem 6.2.1. Moreover, we have, as* $t\to+\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^2}\to 0,\quad \|\overrightarrow{\mathbf{v}}(t)\|_{H^2}\to 0,\quad \|\theta(t)-\overline{\theta}\|_{H^2}\to 0,\quad \|\mathcal{I}(t)\|_{H^3}\to 0.$$

Theorem 6.2.3. *Assume that the initial data* $(\tau_0,\overrightarrow{\mathbf{v}}_0,\theta_0,\mathcal{I}_0)\in\mathcal{H}^4$ *and the compatibility conditions hold. Then there exists a unique global solution* $(\tau(t),\overrightarrow{\mathbf{v}}(t),\theta(t),\mathcal{I}(t))\in L^\infty([0,+\infty),\mathcal{H}^4)$ *to problem (6.2.9)–(6.2.17) verifying for all* $(x,t)\in[0,L]\times[0,+\infty)$,

$$\begin{aligned}&\|\tau(t)-\overline{\tau}\|_{H^4}^2+\|\overrightarrow{\mathbf{v}}(t)\|_{H^4}^2+\|\theta(t)-\overline{\theta}\|_{H^4}^2+\|\mathcal{I}(t)\|_{H^5}^2+\|r(t)-\overline{r}\|_{H^5}^2+\|r_t(t)\|_{H^4}^2+\|\overrightarrow{\mathbf{v}}_t(t)\|_{H^2}^2\\&\quad+\|\theta_t(t)\|_{H^2}^2+\int_0^t\Big(\|\tau-\overline{\tau}\|_{H^4}^2+\|\overrightarrow{\mathbf{v}}\|_{H^5}^2+\|\theta-\overline{\theta}\|_{H^5}^2+\|r-\overline{r}\|_{H^5}^2+\|r_t\|_{H^5}^2\\&\quad+\|\overrightarrow{\mathbf{v}}_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2+\|\overrightarrow{\mathbf{v}}_{tt}\|_{H^1}^2+\|\theta_{tt}\|_{H^1}^2\Big)(s)\,ds\le C_4\end{aligned}$$

where $\overline{\tau},\overline{\theta}>0$ *and* $\overline{r}$ *are also the same as those of theorem 6.2.1. Moreover, we have, as* $t\to+\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^4}\to 0,\quad \|\overrightarrow{\mathbf{v}}(t)\|_{H^4}\to 0,\quad \|\theta(t)-\overline{\theta}\|_{H^4}\to 0,\quad \|\mathcal{I}(t)\|_{H^5}\to 0.$$

6.3 Global Existence and Asymptotic Behavior in $\mathcal{H}^1$

In this section, we establish the global existence and asymptotic behavior for the generalized solutions in $\mathcal{H}^1$ to problem (6.2.9)–(6.2.17) and then complete the proof of theorem 6.2.1 in terms of a series of lemmas.

6.3.1 Global Existence in $\mathcal{H}^1$

We first quote some simple mass conservation and the energy equality.

Lemma 6.3.1. *Under assumptions in theorem 6.2.1, there exists a constant* $C_1 > 0$ *such that the following estimates hold,*

$$\int_0^L \tau(x,t)\,dx = \int_0^L \tau_0(x)\,dx = \tau^* = \frac{b^2 - a^2}{2}, \quad \forall t > 0, \tag{6.3.1}$$

$$\theta(x,t) > 0, \qquad \forall (x,t) \in [0,L] \times [0,+\infty), \tag{6.3.2}$$

$$\begin{aligned}&\int_0^L \left(e + \frac{1}{2}|\overrightarrow{\mathbf{v}}|^2\right)(x,t)\,dx + b\int_0^t \int_0^{+\infty} \int_{S^2 \cap \{\omega \in (0,1)\}} \omega I(L,s,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,ds \\ &- a\int_0^t \int_0^{+\infty} \int_{S^2 \cap \{\omega \in (-1,0)\}} \omega I(0,s,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,ds = \int_0^L \left(e_0 + \frac{1}{2}|\overrightarrow{\mathbf{v}}_0|^2\right)(x)\,dx, \quad \forall t > 0,\end{aligned} \tag{6.3.3}$$

$$\int_0^L (\theta + \theta^{1+d})(x,t)\,dx \le C_1, \quad \forall t > 0. \tag{6.3.4}$$

Proof. Identity (6.3.1) follows directly after integrating (6.2.9) over $Q_t := (0,L) \times (0,t)$ and noting the boundary conditions. Using assumptions (A_1), (6.3.2) follows from the positivity of $\theta_0(x)$ by applying the maximum principle applied to equation (6.2.13).

Integrating (6.2.13) over Q_t and using initial and boundary conditions (6.2.15) and (6.2.17), we can get

$$\int_0^L \left(e + \frac{1}{2}|\overrightarrow{\mathbf{v}}|^2\right)(x,t)\,dx + \int_0^t \int_0^L \tau(S_E)_R\,dx\,ds = \int_0^L \left(e_0 + \frac{1}{2}|\overrightarrow{\mathbf{v}}_0|^2\right)(x)\,dx. \tag{6.3.5}$$

From the definition of $(S_E)_R$ and equation (6.2.16), it follows that

$$\tau(S_E)_R = \left(\int_0^{+\infty} \int_{S^2} \omega r I\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)_x \tag{6.3.6}$$

which, integrated over $(0,L)$ with respect to x, and by noting (6.2.16) and the positivity of radiative intensity I, implies

$$\begin{aligned}\int_0^L \tau(S_E)_R\,dx = b&\int_0^{+\infty} \int_{S^2 \cap \{\omega \in (0,1)\}} \omega I(L,s,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon \\ &- a\int_0^{+\infty} \int_{S^2 \cap \{\omega \in (-1,0)\}} \omega I(0,s,\epsilon,\overrightarrow{\mathbf{\Omega}})\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon \ge 0.\end{aligned} \tag{6.3.7}$$

Inserting (6.3.7) into (6.3.5), we arrive at the identity (6.3.3). Furthermore, inequality (6.3.4) follows from (6.3.3), (6.3.7) and assumption (A_1). □

Now we establish the following entropy-type energy inequalities.

Lemma 6.3.2. *Under assumptions in theorem 6.2.1, the following estimates hold*

$$\int_0^t\int_0^L\left(\frac{\kappa r^2\theta_x^2}{\tau\theta^2}+\frac{\tau\mathcal{Q}}{\theta}\right)dxds+\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\frac{\tau}{\theta}\sigma_a I d\overrightarrow{\Omega}\,d\epsilon dxds\le C_1,\quad \forall t>0, \tag{6.3.8}$$

$$\int_0^t\int_0^L\left(\frac{\tau u^2}{\theta}+\frac{u_x^2+(ru)_x^2}{\tau\theta}+\frac{w_x^2}{\tau\theta}+\frac{(\tau r^{-1}v-rv_x)^2}{\tau\theta}\right)dxds\le C_1,\quad \forall t>0, \tag{6.3.9}$$

$$0<C_\gamma^{-1}\le\int_0^L\theta^\gamma(x,t)dx\le C_\gamma,\ \ \forall\,\gamma\in[0,1],\quad \forall t>0, \tag{6.3.10}$$

where

$$\mathcal{Q}=\frac{1}{\tau^2}\left(\delta(ru)_x^2+\mu((rv)_x^2+r^2w_x^2)\right)-\frac{2\mu}{\tau}(u^2+v^2)_x.$$

Proof. From the second principle of thermodynamics, we know that $\theta\eta_t=e_t+P\tau_t$ with $\eta_\tau=\frac{1}{\theta}(e_\tau-P)$, $\eta_\theta=\frac{e_\theta}{\theta}$, where η represents the total entropy (here the entropy of radiation is zero). Define the free energy $\psi=e-\theta\eta$ with $\psi_\theta=-\eta$, $\psi_\tau=-P$ and let

$$\mathcal{E}(\tau,\theta)=\psi(\tau,\theta)-\psi(1,\theta_0)-(\tau-1)\psi_\tau(1,\theta_0)-(\theta-\theta_0)\psi_\theta(\tau,\theta).$$

Equation (6.2.13) can be rewritten as

$$e_t=\left(\frac{\kappa r^2\theta_x}{\tau}\right)_x+\tau\mathcal{Q}-P(ru)_x-\tau(S_E)_R. \tag{6.3.11}$$

After a direct computation by virtue of equations (6.2.9)–(6.2.12) and (6.3.11), we derive

$$\begin{aligned}&\left(\mathcal{E}+\frac{1}{2}|\overrightarrow{\mathbf{v}}|^2\right)_t+\theta_0\left(\frac{\kappa r^2\theta_x^2}{\tau\theta^2}+\frac{\tau\mathcal{Q}}{\theta}\right)\\&\quad=\left(\left(1-\frac{\theta_0}{\theta}\right)\frac{\kappa r^2\theta_x}{\tau}+\left(\delta\frac{(ru)_x}{\tau}-P+P(1,\theta_0)\right)ru\right)_x\\&\qquad+\mu\left(\frac{rv(rv)_x}{\tau}+\frac{rw(rw)_x}{\tau}-2(u^2+v^2)-w^2\right)_x-\tau\left(1-\frac{\theta_0}{\theta}\right)(S_E)_R.\end{aligned} \tag{6.3.12}$$

From (6.2.14) and the definitions of S and $(S_E)_R$, we derive

$$-\tau\left(1-\frac{\theta_0}{\theta}\right)(S_E)_R = -\int_0^{+\infty}\int_{S^2}\omega(rI)_x d\overrightarrow{\mathbf{\Omega}}\,d\epsilon + \tau\frac{\theta_0}{\theta}\int_0^{+\infty}\int_{S^2}\sigma_a(B-I)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon. \tag{6.3.13}$$

Using assumptions (A_1) and (A_8), we have

$$\tau\frac{\theta_0}{\theta}\int_0^{+\infty}\int_{S^2}\sigma_a B d\overrightarrow{\mathbf{\Omega}}\,d\epsilon \le C_1(1+\theta^\alpha) \le C_1 e. \tag{6.3.14}$$

Inserting (6.3.13) and (6.3.14) into (6.3.12), integrating the result over Q_t and noting that

$$\tau\mathcal{Q} \ge C_1\left(\tau u^2 + \frac{u_x^2+(ru)_x^2}{\tau} + \frac{w_x^2}{\tau} + \frac{(\tau r^{-1}v - rv_x)^2}{\tau}\right) \ge 0, \tag{6.3.15}$$

and using the initial boundary conditions (6.2.15)–(6.2.17) and (6.3.3) in lemma 6.3.1, we derive

$$\int_0^L\left(\mathcal{E}+\frac{1}{2}|\overrightarrow{\mathbf{v}}|^2\right)dx + \int_0^t\int_0^L\theta_0\left(\frac{\kappa r^2\theta_x^2}{\tau\theta^2}+\frac{\tau\mathcal{Q}}{\theta}+\int_0^{+\infty}\int_{S^2}\frac{\tau}{\theta}\sigma_a I d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)dxds \le C_1. \tag{6.3.16}$$

Now by the same arguments as in lemma 3.3.2, we can obtain (see also [31])

$$\mathcal{E} \ge \frac{1}{4}C_1^{-1}(\theta+\theta^{1+d}) - C_1$$

which, along with (6.3.4), leads to estimate (6.3.8). Furthermore, in view of (6.3.15) and (6.3.8), estimate (6.3.9) holds. By the Jensen inequality and the Young inequality, we conclude that (6.3.10) holds. □

The next lemma will be used to get the upper bound of the specific volume which plays an important role in removing the smallness of initial data. For our convenience, we introduce the notations

$$\theta_{\max}(t) = \max_{x\in[0,L]}\theta(x,t), \qquad \tau_{\max}(t) = \max_{x\in[0,L]}\tau(x,t).$$

Lemma 6.3.3. *Under assumptions in theorem 6.2.1, we have, for all* $(x,t)\in[0,L]\times[0,+\infty)$,

$$\int_0^t \|v(s)\|^2 ds \le C_1, \tag{6.3.17}$$

$$\theta_{\max}^{1+d}(t) \le C_1\left(1+\tau_{\max}(t)\int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dx\right), \tag{6.3.18}$$

$$\theta^{1+d}(x,t) \ge C - C\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx. \tag{6.3.19}$$

Proof. By virtue of (6.2.18), we can rewrite (6.2.11) as

$$(rv)_t = \mu r^2\left(\frac{(rv)_x}{\tau}\right)_x. \tag{6.3.20}$$

Multiplying (6.3.20) by rv over $[0, L]$ and using an integration by parts, and (6.2.15) and (6.2.18), we get

$$\left(\frac{1}{2}\int_0^L (rv)^2\,dx\right)_t = -\mu\int_0^L (r^3v)_x\frac{(rv)_x}{\tau}\,dx = -\mu\int_0^L \frac{r^2(rv)_x^2}{\tau}dx. \tag{6.3.21}$$

Integrating (6.3.21) with respect to t, we easily see

$$\int_0^L (rv)^2\,dx + 2\mu\int_0^t\int_0^L \frac{r^2(rv)_x^2}{\tau}dxds = \int_0^L (r_0v_0)^2\,dx. \tag{6.3.22}$$

By the Poincaré inequality, the Hölder inequality, (6.3.1) and (6.3.22), we can derive

$$\begin{aligned}\int_0^t\int_0^L v^2\,dxds &\le C\int_0^t \max_{x\in[0,L]}|rv|^2\,ds \le C\int_0^t\left(\int_0^L |(rv)_x|\,dx\right)^2 ds\\ &\le C\int_0^t\left(\int_0^L \tau\,dx\right)\left(\int_0^L \frac{(rv)_x^2}{\tau}dx\right)ds\\ &\le C\int_0^t\int_0^L \frac{r^2(rv)_x^2}{\tau}dxds \le C_1.\end{aligned}$$

It follows from (6.3.10) that there exists a point $x_0(t) \in [0, L]$ such that

$$0 < C_1^{-1} \le \int_0^L \theta(x,t)\,dx = \theta(x_0(t),t) \le C_1.$$

By Sobolev's embedding theorem and the Hölder inequality, we have

$$\left|\theta^{\frac{1+d}{2}}(x,t) - \theta^{\frac{1+d}{2}}(x_0(t),t)\right| \le C_1\left|\int_{x_0(t)}^x \theta^{\frac{d-1}{2}}\theta_x(y,t)\,dy\right| \le C_1\int_0^L \theta^{\frac{d-1}{2}}|\theta_x|\,dx. \tag{6.3.23}$$

Noting assumption (A_6) and (6.3.4), by the Hölder inequality, we derive from (6.3.23) that

$$\begin{aligned}\theta^{1+d}(x,t) &\le C_1 + \left(\int_0^L \theta^{\frac{d-1}{2}}|\theta_x|\,dx\right)^2\\ &\le C_1 + \int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dx\int_0^L \frac{\tau\theta^{3-d}}{\kappa r^2}dx\\ &\le C_1 + C_1\tau_{\max}(t)\int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dx\int_0^L (1+\theta)^{1+d}dx\\ &\le C_1 + C_1\tau_{\max}(t)\int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dx.\end{aligned}$$

Thus, we obtain (6.3.18).

In the same manner, by the Hölder inequality, we can also derive from (6.3.23) that

$$\begin{aligned}\theta^{1+d}(x,t) &\ge C_1^{-1} - \left(\int_0^L \theta^{\frac{d-1}{2}}|\theta_x|\,dx\right)^2\\ &\ge C_1^{-1} - \left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)\left(\int_0^L \frac{\theta^{3-d}}{\kappa r^2}dx\right)\\ &\ge C_1^{-1} - C_1\left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)\end{aligned}$$

which implies (6.3.19). □

The next lemma gives the upper bound of the volume, which plays a key role in obtaining the main results.

Lemma 6.3.4. *Under assumptions in theorem 6.2.1, the following estimate holds for all* $(x,t)\in[0,L]\times[0,+\infty)$,

$$\tau(x,t)\le C_1. \tag{6.3.24}$$

Proof. First, similarly to the proof of lemma 2.3.2 (see also lemma 5.2.2 in Qin [99]), we can easily derive that there exists a point $x_1(t)\in[0,L]$ such that for each $t\ge 0$, the function $\tau(x,t)$ can be expressed as

$$\tau(x,t) = D(x,t)\left(\frac{1}{B(x,t)} + \frac{1}{\delta}\int_0^t \frac{P\tau}{D(x,s)}\frac{B(x,s)}{B(x,t)}ds\right) \tag{6.3.25}$$

where

$$D(x,t) = \tau_0(x)\exp\left\{\frac{1}{\delta}\left(\frac{1}{\tau^*}\int_0^L \tau_0\left(\int_0^x \frac{u_0}{r_0}dy\right)dx + \int_{x_1(t)}^x \frac{u}{r}\,dy - \int_0^x \frac{u_0}{r_0}dy\right)\right\}, \tag{6.3.26}$$

$$\begin{aligned}B(x,t) = \exp\Bigg\{&\frac{1}{\delta\tau^*}\int_0^t\int_0^L\left(\frac{u^2+v^2}{2}+P\tau\right)dxds + \frac{a^2}{2\delta\tau^*}\int_0^t\int_0^L \frac{u^2-v^2}{r^2}dxds\\ &+\frac{1}{\delta}\int_0^t\int_x^L \frac{u^2-v^2}{r^2}dyds\Bigg\}.\end{aligned} \tag{6.3.27}$$

Let

$$M(x,t) = \frac{1}{\delta\tau^*}\int_0^L \left(\frac{u^2+v^2}{2} + P\tau\right)dx + \frac{a^2}{2\delta\tau^*}\int_0^L \frac{u^2-v^2}{r^2}dx + \frac{1}{\delta}\int_x^L \frac{u^2-v^2}{r^2}dy. \tag{6.3.28}$$

Then it follows from assumption (A_4), (6.3.3), (6.3.4) and (6.3.28) that

$$\begin{aligned} M(x,t) &\geq C_1^{-1}\int_0^L P\tau\,dx - C_1\int_0^L v^2\,dx \\ &\geq C_1^{-1}\int_0^L (1+\theta^{1+d})\,dx - C_1\int_0^L v^2\,dx \\ &\geq C_1^{-1} - C_1\int_0^L v^2\,dx. \end{aligned} \tag{6.3.29}$$

For any fixed s, t satisfying $0 \leq s \leq t$, we derive from (6.3.17) and (6.3.28) that

$$-\int_s^t M(x,\xi)\,d\xi \leq -C_1(t-s) + C_1\int_s^t\int_0^L v^2\,dxd\xi \leq C_1 - C_1(t-s). \tag{6.3.30}$$

Thus it follows from (6.3.27), (6.3.28) and (6.3.30) that

$$\frac{B(x,s)}{B(x,t)} = \exp\left(-\int_s^t M(x,\xi)\,d\xi\right) \leq C_1\exp(-C_1(t-s)). \tag{6.3.31}$$

Finally, we derive from (6.3.3) that

$$C_1^{-1} \leq D(x,t) \leq C_1$$

which, combined with (6.3.1), (6.3.18), (6.3.25) and (6.3.31), yields that

$$\begin{aligned} \tau(x,t) &\leq C_1 + C_1\int_0^t P\tau\exp(-C_1(t-s))\,ds \\ &\leq C_1 + C_1\int_0^t (1+\theta^{1+d})\exp(-C_1(t-s))\,ds \\ &\leq C_1 + C_1\int_0^t \exp(-C_1(t-s))\,ds + C_1\int_0^t \theta^{1+d}\exp(-C_1(t-s))\,ds \\ &\leq C_1 + C_1\int_0^t \tau_{\max}(s)\int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dxds. \end{aligned} \tag{6.3.32}$$

Thus we can obtain

$$\tau_{\max}(t) \leq C_1 + C_1\int_0^t \tau_{\max}(s)\int_0^L \frac{\kappa r^2\theta_x^2}{\tau\theta^2}dxds. \tag{6.3.33}$$

Applying the Gronwall inequality, we have

$$\tau_{\max}(t) \leq C_1. \tag{6.3.34}$$

Therefore, we complete the proof. □

Lemma 6.3.5. *Under assumptions in theorem 6.2.1, the following estimate holds for all* $(x,\, t) \in [0,\, L] \times [0,\, +\infty)$,

$$\tau(x, t) \geq C_1 > 0. \tag{6.3.35}$$

Proof. Inserting (6.2.9) into (6.2.10) and recalling (6.2.18), we know

$$\left(\frac{u}{r}\right)_t = \delta(\log\tau)_{xt} - P_x + \frac{v^2 - u^2}{r^2}.$$

Let

$$\Gamma = \delta\frac{(ru)_x}{\tau} - P = \delta(\log\tau)_t - P,$$

then

$$\left(\frac{u}{r}\right)_t = \Gamma_x + \frac{v^2 - u^2}{r^2}.$$

Integrating the above equation over $[s,\, t] \times [x,\, L]$, for any fixed s and t satisfying $0 \leq s \leq t$, we have for any $0 \leq x \leq L$,

$$\begin{aligned}
&\int_L^x \left(\frac{u}{r}(y, t) - \frac{u}{r}(y, s)\right) dy + \int_s^t \int_L^x \frac{u^2 - v^2}{r^2} dy d\xi \\
&= \int_s^t \Gamma(x, \xi)\, d\xi - \int_s^t \Gamma(L, \xi)\, d\xi \\
&= \delta \log\tau(x, t) - \delta\log\tau(x, s) - \int_s^t P d\xi - \int_s^t \Gamma(L, \xi)\, d\xi.
\end{aligned} \tag{6.3.36}$$

Denote

$$F(s, t) = \int_s^t \Gamma(L, \xi)\, d\xi$$

and

$$A(x; s, t) = \int_L^x \left(\frac{u}{r}(y, t) - \frac{u}{r}(y, s)\right) dy + \int_s^t \int_L^x \frac{u^2 - v^2}{r^2} dy d\xi,$$

we derived from (6.3.36) that

$$\frac{d}{dt}\left(\tau(x, t) \exp\left(-\frac{1}{\delta}(A + F)\right)\right) = \frac{P\tau}{\delta}\exp\left(-\frac{1}{\delta}(A + F)\right). \tag{6.3.37}$$

Thus, we obtain

$$\tau(x,t)\exp\left(-\frac{1}{\delta}(A+F)\right)=\tau(x,s)+\int_s^t\frac{P\tau}{\delta}\exp\left(-\frac{1}{\delta}(A+F)\right)d\xi. \tag{6.3.38}$$

Using (6.3.9), (6.3.24), the Poincaré inequality and the Hölder inequality, we have

$$\begin{aligned}\int_0^t\int_0^L\frac{u^2}{r^2}dxds&\le C_1\int_0^t\max_{x\in[0,L]}|u|^2ds\le C_1\int_0^t\left(\int_0^L|u_x|dx\right)^2ds\\&\le C_1\int_0^t\left(\int_0^L\theta dx\right)\left(\int_0^L\frac{u_x^2}{\theta}dx\right)ds\\&\le C_1\int_0^t\int_0^L\frac{u_x^2}{\tau\theta}dxds\le C_1.\end{aligned} \tag{6.3.39}$$

Thus using (6.3.3), (6.3.17) and (6.3.39), we get

$$|A(x;s,t)|\le C_1$$

which, combining with (6.3.38), leads to

$$\begin{aligned}&C_1^{-1}\left(\tau(x,s)+\int_s^tP\tau(x,\xi)\exp\left(-\frac{1}{\delta}F(s,\xi)\right)d\xi\right)\\&\le\tau(x,t)\exp\left(-\frac{1}{\delta}F(s,t)\right)\\&\le C_1\left(\tau(x,s)+\int_s^tP\tau(x,\xi)\exp\left(-\frac{1}{\delta}F(s,\xi)\right)d\xi\right).\end{aligned} \tag{6.3.40}$$

Integrating (6.3.40) with respect to x over $[0,L]$ and assumption (A_4) and (6.3.4), we have

$$\begin{aligned}C_1^{-1}\left(1+\int_s^t\exp\left(-\frac{1}{\delta}F(s,\xi)\right)d\xi\right)&\le\exp\left(-\frac{1}{\delta}F(s,t)\right)\\&\le C_1\left(1+\int_s^t\exp\left(-\frac{1}{\delta}F(s,\xi)\right)d\xi\right).\end{aligned} \tag{6.3.41}$$

Then, by Gronwall inequality, we derive, for $0\le s\le t$,

$$C_1^{-1}e^{C_1^{-1}(t-s)}\le\exp\left(-\frac{1}{\delta}F(s,t)\right)\le C_1e^{C_1(t-s)}. \tag{6.3.42}$$

Setting $s=0$ in the left-hand side of (6.3.40) while, using (6.3.19), (6.3.41) and (6.3.42), we get

$$\begin{aligned}
\tau(x,t) &\geq C_1 \exp\left(\frac{1}{\delta}F(0,t)\right)\left(\tau(x,0) + \int_0^t P\tau(x,s)\exp\left(-\frac{1}{\delta}F(0,s)\right)ds\right) \\
&\geq C_1 \exp\left(\frac{1}{\delta}F(0,t)\right)\left(1 + \int_0^t \exp\left(-\frac{1}{\delta}F(0,s)\right)ds\right. \\
&\quad \left. + \int_0^t \theta^{1+d}(x,s)\exp\left(-\frac{1}{\delta}F(0,s)\right)ds\right) \\
&\geq C_1\left(\exp\left(\frac{1}{\delta}F(0,t)\right) + \int_0^t \exp\left(\frac{1}{\delta}F(s,t)\right)ds\right. \\
&\quad \left. - \int_0^t \exp\left(\frac{1}{\delta}F(s,t)\right)\left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)ds\right) \\
&\geq C_1^{-1}\exp(-C_1^{-1}t) + \int_0^t C_1^{-1}\exp(-C_1^{-1}(t-s))\,ds \\
&\quad - C_1\int_0^t \exp\left(\frac{1}{\delta}F(s,t)\right)\left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)ds \\
&\geq C_1 - C_1\int_0^t \exp\left(\frac{1}{\delta}F(s,t)\right)\left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)ds.
\end{aligned} \tag{6.3.43}$$

By virtue of lemma 4.3.4, we know

$$\lim_{t\to+\infty}\int_0^t \exp\left(\frac{1}{\delta}F(s,t)\right)\left(\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dx\right)ds = \lim_{t\to+\infty}\int_t^{t+1}\int_0^L \frac{\kappa r^2\theta_x^2}{\theta^2}dxds = 0. \tag{6.3.44}$$

Therefore, there exists a time $T > 0$ satisfying for all $t \geq T$,

$$\tau(x,t) \geq C_1. \tag{6.3.45}$$

Finally, if $0 \leq t \leq T$, it follows from (6.3.40) and (6.3.42) that

$$\begin{aligned}
\tau(x,t) &\geq C_1 \exp\left(\frac{1}{\delta}F(0,t)\right)\left(\tau(x,0) + \int_0^t P\tau(x,s)\exp\left(-\frac{1}{\delta}F(0,s)\right)ds\right) \\
&\geq C_1 \exp\left(\frac{1}{\delta}F(0,t)\right)\tau(x,0) \geq C_1 e^{-C_1 t} \geq C_1 e^{-C_1 T}.
\end{aligned} \tag{6.3.46}$$

Thus there exists a constant $C_1 > 0$ such that for any $t > 0$,

$$\tau(x,t) \geq C_1^{-1} > 0$$

which, along with (6.3.45), leads to (6.3.35). □

By Sobolev's embedding theorem and the Hölder inequality, we easily derive the following corollary from lemmas 6.3.3–6.3.5. This corollary plays an important role in obtaining our main results.

Corollary 6.3.1. *Under assumptions in theorem 6.2.1, the following estimate holds for all* $(x,t) \in [0,L] \times [0,+\infty)$ *and* $0 \leq m \leq \frac{q+d+1}{2}$,

$$C_1^{-1} - C_1 V_1(t) \leq \theta^{2m}(x,t) \leq C_1 + C_1 V_1(t) \tag{6.3.47}$$

where $V_1(t) = \int_0^L \frac{\kappa\theta_x^2}{\theta^2}dx$ *verifying* $\int_0^{+\infty} V_1(t)\,dt \leq C_1$.

Lemma 6.3.6. *Under assumptions in theorem 6.2.1, the following estimates hold for any $t > 0$ and $\beta \in (0, 1]$,*

$$\int_0^t \int_0^L \left(\frac{\kappa r^2 \theta_x^2}{(1+\theta)^{1+\beta}} + \frac{\tau \mathcal{Q}}{(1+\theta)^{\beta}} \right) dxds \leq C_1, \tag{6.3.48}$$

$$\int_0^t \|u(s)\|_{L^\infty}^2 ds \leq C_1, \quad \int_0^t \left(\|u_x\|^2 + \|w_x\|^2 \right)(s)\, ds \leq C_1 \left(1 + \sup_{0 \leq s \leq t} \|\theta(s)\|_{L^\infty} \right)^{\beta}. \tag{6.3.49}$$

Proof. We prove this lemma by adopting the main idea from lemma 5.2.3 in Qin [99] and lemma 2.4 in Qin [98]. We only give the proof of (6.3.48) for $\beta \in (0, 1)$, because the case of $\beta = 1$ is the obvious result of (6.3.8). Multiplying (6.3.11) by $(1+\theta)^{-\beta} \left(\int_0^L (1+\theta)^{1+d-\beta} dx \right)^{-1}$ in $L^2(0, L)$, we find

$$\begin{aligned}
&\left(\int_0^L (1+\theta)^{1+d-\beta} dx \right)^{-1} \int_0^L \left(\frac{\beta \kappa r^2 \theta_x^2}{(1+\theta)^{1+\beta}} + \frac{\tau \mathcal{Q}}{(1+\theta)^{\beta}} \right) dx \\
&\quad = C_1 \beta L^{-1} \frac{d}{dt} \int_0^L \log \tau \, dx + \left(\int_0^L (1+\theta)^{1+d-\beta} dx \right)^{-1} \left(\int_0^L (1+\theta)^{-\beta} (e_t + P(ru)_x) dx \right) \\
&\qquad + \left(\int_0^L (1+\theta)^{1+d-\beta} dx \right)^{-1} \int_0^L (1+\theta)^{-\beta} \tau (S_E)_R dx - C_1 \beta L^{-1} \int_0^L \frac{(ru)_x}{\tau} dx.
\end{aligned} \tag{6.3.50}$$

Applying assumptions (A_1) and (A_3), we have

$$\begin{aligned}
&\int_0^L (1+\theta)^{-\beta} (e_t + P(ru)_x) dx \\
&\quad = \int_0^L (1+\theta)^{-\beta} e_\theta \theta_t dx + \int_0^L (1+\theta)^{-\beta} \theta P_\theta (ru)_x dx \\
&\quad \leq \frac{C_1}{1+d-\beta} \int_0^L (1+\theta)^{1+d-\beta} dx \frac{d}{dt} \log \int_0^L (1+\theta)^{1+d-\beta} dx \\
&\qquad + C_1 \beta \int_0^L (1+\theta)^{1+d-\beta} \frac{(ru)_x}{\tau} dx.
\end{aligned} \tag{6.3.51}$$

Noting the positivity of θ and (6.3.5), we find

$$\int_0^t \int_0^L (1+\theta)^{-\beta} \tau (S_E)_R dxds \leq C_1. \tag{6.3.52}$$

Integrating (6.3.50) with respect to t and then using (6.3.51) and (6.3.52), lemmas 6.3.1, 6.3.2, 6.3.4, 6.3.6 and the Young inequality, we have, for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^t\int_0^L\left(\frac{\kappa r^2\theta_x^2}{(1+\theta)^{1+\beta}}+\frac{\tau\mathcal{Q}}{(1+\theta)^{\beta}}\right)dxds\\
&\le C_1+C_1\int_0^t\int_0^L\frac{(ru)_x}{\tau}\left((1+\theta)^{1+d-\beta}-L^{-1}\int_0^L(1+\theta)^{1+d-\beta}dx\right)dxds\\
&\le C_1+C_1\int_0^t\left(\int_0^L\frac{(ru)_x^2}{\theta}dx\right)^{\frac12}\left(\int_0^L\theta dx\right)^{\frac12}\left\|(1+\theta)^{1+d-\beta}-L^{-1}\int_0^L(1+\theta)^{1+d-\beta}dx\right\|_{L^\infty}ds\\
&\le C_1+C_1(\varepsilon)\int_0^t\int_0^L\frac{(ru)_x^2}{\theta}dxds+\varepsilon\int_0^t\int_0^L\frac{\theta_x^2}{(1+\theta)^{1+\beta}}dx\int_0^L(1+\theta)^{1+d-\beta}dxds\\
&\le C_1(\varepsilon)+C_1\varepsilon\int_0^t\int_0^L\frac{\kappa\theta_x^2}{(1+\theta)^{1+\beta}}dxds
\end{aligned}$$

which, by picking $\varepsilon>0$ small enough, yields (6.3.48). By (6.3.48) and lemmas 6.3.1–6.3.5, we have

$$\int_0^t\|u(s)\|_{L^\infty}^2ds\le\int_0^t\left(\int_0^L\frac{u_x^2}{\theta}dx\right)\left(\int_0^L\theta dx\right)ds\le C_1,$$

$$\begin{aligned}
\int_0^t\left(\|u_x\|^2+\|w_x\|^2\right)(s)ds&\le\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta}\int_0^t\int_0^L\frac{u_x^2+w_x^2}{(1+\theta)^{\beta}}dxds\\
&\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\beta}.
\end{aligned}$$

Thus we complete the proof. □

Employing corollary 6.3.1 and lemmas 6.3.1 and 6.3.6, we conclude the next corollary.

Corollary 6.3.2. *Under assumptions in theorem 6.2.1, the following estimate holds for any* $(x,t)\in[0,L]\times[0,+\infty)$ *and* $0\le m\le\frac{q+d+1}{2}$,

$$\int_0^t\int_0^L(1+\theta^{2m})u^2(x,t)\le C_1.$$

Lemma 6.3.7. *Under assumptions in theorem 6.2.1, the following estimates hold for any* $t>0$ *and* $\beta\in(0,1]$,

$$\|v(t)\|_{L^\infty}^2+\|v(t)\|_{H^1}^2+\int_0^t\left(\|v_t\|^2+\|v\|_{H^1}^2+\|v\|_{L^\infty}^2\right)(s)ds\le C_1, \tag{6.3.53}$$

$$\|w(t)\|_{L^\infty}^2+\|w(t)\|_{H^1}^2+\int_0^t\left(\|w_t\|^2+\|w\|_{H^1}^2+\left\|\left(\frac{(rw)_x}{\tau}\right)_x\right\|^2\right)(s)ds\le C_1, \tag{6.3.54}$$

$$\|\tau_x(t)\| + \int_0^t\int_0^L (1+\theta)^{1+d}\tau_x^2(x,s)\,dxds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\beta}. \tag{6.3.55}$$

Proof. Similarly to the proof of lemma 4.3.6, we can obtain (6.3.53) and (6.3.54). Similarly to the proof of lemma 2.3.5, we can also prove (6.3.55). See also lemma 5.2.4 in Qin [99]. □

Lemma 6.3.8. *Under assumptions in theorem 6.2.1, the following estimates hold for any* $t>0$ *and* $\beta\in(0,1]$,

$$\begin{aligned}&\|u(t)\|_{H^1}^2 + \int_0^t\left(\|u_t\|^2 + \|u\|_{H^1}^2 + \|u\|_{L^\infty}^2 + \left\|\left(\frac{(ru)_x}{\tau}\right)_x\right\|^2\right)(s)\,ds\\ &\quad\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\max\{\beta,d+2+\beta-q\}},\end{aligned} \tag{6.3.56}$$

$$\|u(t)\|_{L^\infty} \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac14\max\{\beta,d+2+\beta-q\}}. \tag{6.3.57}$$

Proof. Applying equation (6.2.9) and (6.2.10), assumptions (A_2) and (A_3), corollary 6.3.1, lemma 6.3.6 and (6.3.55), we can easily derive

$$\begin{aligned}&\frac{\delta}{2}\frac{d}{dt}\int_0^L\frac{(ru)_x^2}{\tau}dx + \|u_t\|^2\\ &= \frac{\delta}{2}\int_0^L ru\left(\frac{(ru)_x^2}{\tau^2}\right)_x dx - \delta\int_0^L u^2\left(\frac{(ru)_x}{\tau}\right)_x dx - \int_0^L P_x u_t\,dx + \int_0^L\frac{u^2}{r}u_t\,dx\\ &\le \int_0^L\left(\frac{ru(ru)_x}{\tau} - u^2\right)\left(\frac{u_t}{r} + P_x - \frac{v^2}{r^2}\right)dx + \frac14\|u_t\|^2 + C_1(1+V_1(t))\|\tau_x\|^2\\ &\quad + C_1\|v\|_{L^\infty}^2 + C_1\big(1+\|\theta(t)\|_{L^\infty}\big)^{\max\{d+2+\beta-q,0\}}\int_0^L\frac{\kappa\theta_x^2}{(1+\theta)^{1+\beta}}dx\\ &\le \frac12\|u_t\|^2 + C_1\|v\|_{L^\infty}^2 + C_1\|u\|_{L^\infty}^2 + C_1\Big(\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2\Big)\int_0^L\frac{(ru)_x^2}{\tau}dx\\ &\quad + C_1(1+V_1(t))\|\tau_x\|^2 + C_1\big(1+\|\theta(t)\|_{L^\infty}\big)^{\max\{d+2+\beta-q,0\}}\int_0^L\frac{\kappa\theta_x^2}{(1+\theta)^{1+\beta}}dx\end{aligned}$$

which, by the Gronwall inequality, (6.3.55) and the embedding theorem, yields (6.3.56). By the interpolation inequality and lemma 6.3.1, we have

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le C_1\|u(t)\|^{\frac12}\|u_x(t)\|^{\frac12} + C_1\|u(t)\| \le C_1 + C_1\|u_x(t)\|^{\frac12}\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac14\max\{\beta,d+2+\beta-q\}}.\end{aligned}$$

Thus we complete the proof. □

The next lemma concerns the estimates on radiative density I.

Lemma 6.3.9. *Under assumptions in theorem 6.2.1, the following estimates hold for any* $t > 0$,

$$\int_0^{+\infty}\int_0^L\int_{S^2}\tau(\sigma_a+\sigma_s)I^2\,d\overrightarrow{\Omega}\,dxd\epsilon+\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxd\epsilon \\ \leq C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t), \tag{6.3.58}$$

$$\int_0^{+\infty}\int_{S^2}I(x,t,\overrightarrow{\Omega},\epsilon)\,d\overrightarrow{\Omega}\,d\epsilon\leq C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t). \tag{6.3.59}$$

Proof. Multiplying (6.2.14) by rI and then integrating the resultant over $[0, L] \times S^2$ and using the boundary conditions (6.2.16), we obtain

$$\frac{1}{2}\int_{S^2}\omega(rI)^2\big|_0^L\,d\overrightarrow{\Omega}+\int_0^L\int_{S^2}\tau(\sigma_a rI^2+\sigma_s r(\widetilde{I}-I)^2)\,d\overrightarrow{\Omega}\,dx \\ =\int_0^L\int_{S^2}\tau\sigma_a BI\,d\overrightarrow{\Omega}\,dx+\int_0^L\int_{S^2}\tau\sigma_s r(\widetilde{I}-I)\widetilde{I}\,d\overrightarrow{\Omega}\,dx. \tag{6.3.60}$$

Noting the boundary condition (6.2.16), we know

$$\int_0^{+\infty}\int_{S^2}\omega(b^2I^2(L,t;\overrightarrow{\Omega},\epsilon)-a^2I^2(0,t;\overrightarrow{\Omega},\epsilon))\,d\overrightarrow{\Omega}\,d\epsilon \\ =\int_0^{+\infty}\int_{S^2\cap\{\omega\in(0,1)\}}\omega b^2I^2(L,t;\overrightarrow{\Omega},\epsilon)\,d\overrightarrow{\Omega}\,d\epsilon \\ -\int_0^{+\infty}\int_{S^2\cap\{\omega\in(-1,0)\}}\omega a^2I^2(0,t;\overrightarrow{\Omega},\epsilon)\,d\overrightarrow{\Omega}\,d\epsilon \\ \geq 0. \tag{6.3.61}$$

Integrating (6.3.60) with respect to frequency ϵ over $[0, +\infty)$, using the assumption (A_8) and Young inequality, we have

$$\frac{1}{2}\int_0^{+\infty}\int_{S^2}\omega(b^2I^2(L,t;\overrightarrow{\Omega},\epsilon)-a^2I^2(0,t;\overrightarrow{\Omega},\epsilon))\,d\overrightarrow{\Omega}\,d\epsilon \\ +\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_aI^2\,d\overrightarrow{\Omega}\,dxd\epsilon+\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,dxd\epsilon \\ \leq\frac{1}{2}\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_aI^2\,d\overrightarrow{\Omega}\,dxd\epsilon+C_1\int_0^{+\infty}\int_0^L\int_{S^2}\theta^{1+\alpha}f(\epsilon,\overrightarrow{\Omega})\,d\overrightarrow{\Omega}\,dxd\epsilon \\ \leq\frac{1}{2}\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_aI^2\,d\overrightarrow{\Omega}\,dxd\epsilon+C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t)$$

which, together with (6.3.61), implies

$$\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_a I^2 d\overrightarrow{\Omega}\,dxd\epsilon+\int_0^{+\infty}\int_0^L\int_{S^2}\tau\sigma_s(\widetilde{I}-I)^2 d\overrightarrow{\Omega}\,dxd\epsilon\le C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t).$$

Similarly, we also derive from (6.3.60) by the Young inequality and assumptions (A_8) that

$$\int_0^{+\infty}\int_0^L\int_{S^2}\tau(\sigma_a+\sigma_s)I^2 d\overrightarrow{\Omega}\,dxd\epsilon\le C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t).$$

Thus, we obtain estimate (6.3.58).

Now we consider the following integro-differential equation

$$\begin{cases}\omega(rI)_x=\tau S=\tau(\sigma_a B+\sigma_s\widetilde{I})-\tau(\sigma_a+\sigma_s)I,\\ I(0,t;\epsilon,\overrightarrow{\Omega})=0\quad\text{for }\ \omega\in(0,1),\\ I(L,t;\epsilon,\overrightarrow{\Omega})=0\quad\text{for }\ \omega\in(-1,0),\\ I(x,0;\epsilon,\overrightarrow{\Omega})=I_0(x,\epsilon,\overrightarrow{\Omega}).\end{cases}\tag{6.3.62}$$

Solving the above ordinary differential equation and using the boundary conditions, we arrive at

$$I(x,t;\epsilon,\overrightarrow{\Omega})=\begin{cases}\dfrac{1}{r}\displaystyle\int_0^x\exp\left(\int_x^y\frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)\frac{\tau}{\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dy, & \text{for }\omega\in(0,1),\\ -\dfrac{1}{r}\displaystyle\int_x^L\exp\left(\int_x^y\frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)\frac{\tau}{\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dy, & \text{for }\omega\in(-1,0).\end{cases}\tag{6.3.63}$$

Using the Young inequality and (6.3.58), we have for $\omega\in(0,1)$,

$$\begin{aligned}\int_0^{+\infty}\int_{S^2}I\,d\overrightarrow{\Omega}\,d\epsilon&=\int_0^{+\infty}\int_{S^2}\frac{1}{r}\int_0^x\exp\left(\int_x^y\frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)\frac{\tau}{\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dyd\overrightarrow{\Omega}\,d\epsilon\\ &\le\left|\int_0^{+\infty}\int_{S^2}\frac{1}{r}\int_0^L\frac{\tau}{\omega}(\sigma_a B+\sigma_s\widetilde{I})\,dxd\overrightarrow{\Omega}\,d\epsilon\right|\\ &\le C_1\int_0^L\theta^{1+\alpha}dx+C_1\left|\int_0^L\int_0^{+\infty}\int_{S^2}\sigma_s\left(\frac{1}{\omega^2}+(\widetilde{I}-I)^2+I^2\right)d\overrightarrow{\Omega}\,d\epsilon dx\right|\\ &\le C_1\sup_{x\in[0,L]}\theta^{\max\{\alpha-d,0\}}(x,t).\end{aligned}\tag{6.3.64}$$

Analogously, we have the same result for $\omega\in(-1,0)$. So we complete the proof of this lemma. □

Now we turn to estimate the temperature θ.

Lemma 6.3.10. *Under assumptions in theorem 6.2.1, the following estimate holds for any* $t > 0$ *and* $\beta \in (0, 1]$,

$$\left\|\theta+\theta^{1+d}\right\|^2+\int_0^t\int_0^L(1+\theta^{q+d})\theta_x^2dxds\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+\beta}. \tag{6.3.65}$$

Proof. Multiplying (6.3.11) by e over Q_t, we get

$$\begin{aligned}&\left\|\theta+\theta^{1+d}\right\|^2+\int_0^t\int_0^L(1+\theta^{q+d})\theta_x^2dxds\\ &\quad\le C_1+C_1\int_0^t\int_0^L\left|\tau\mathcal{Q}e+ru(eP)_x-\frac{\kappa\theta_xe_x\tau_x}{\tau}\right|dxds+\left|\int_0^t\int_0^Le\tau(S_E)_Rdxds\right|.\end{aligned} \tag{6.3.66}$$

Now we estimate each term on the right-hand side of (6.3.59). Using the embedding theorem, the Hölder inequality and lemmas 6.3.1–6.3.8, we have

$$\begin{aligned}\int_0^t\int_0^L|\tau\mathcal{Q}e|dxds&\le C_1\int_0^t\int_0^L|(ru)_x^2e|dxds+C_1\int_0^t\int_0^L|((rv)_x^2+r^2w_x^2)e|dxds\\ &\quad+C_1\int_0^t\int_0^L|(u^2+v^2)_xe|dxds\\ &\le C_1\int_0^t\Big(\left\|(ru)_x\right\|_{H^1}^2+\left\|(rv)_x\right\|_{H^1}^2+\|w_x\|_{H^1}^2+\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2\Big)ds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\max\{\beta,d+2+\beta-q\}}\end{aligned} \tag{6.3.67}$$

and

$$\begin{aligned}&\int_0^t\int_0^L|ru(eP)_x-\kappa\theta_xe_x\tau_x|dxds\\ &\quad\le C_1\int_0^t\int_0^L(1+\theta^{2(1+d)})|u\tau_x|dxds+C_1\int_0^t\int_0^L(1+\theta^{1+2d})|u\theta_x|dxds\\ &\qquad+C_1\int_0^t\int_0^L(1+\theta^{1+q+d})|\tau_x\theta_x|dxds\\ &\quad\le C_1\Big(\int_0^t\int_0^L(1+\theta^{2(1+d)})\tau_x^2dxds\Big)^{\frac12}\Big(\int_0^t\int_0^L(1+\theta^{2(1+d)})u^2dxds\Big)^{\frac12}\\ &\qquad+C_1\int_0^t\int_0^L(1+\theta^{2+3d-q})u^2dxds+\frac12\int_0^t\int_0^L(1+\theta^{q+d})\theta_x^2dxds\\ &\qquad+C_1\int_0^t\int_0^L(1+\theta^{2+q+d})\tau_x^2dxds\\ &\quad\le\frac12\int_0^t\int_0^L(1+\theta^{q+d})\theta_x^2dxds+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{1+\beta}.\end{aligned} \tag{6.3.68}$$

At last, we use corollary 6.3.1 and lemmas 6.3.1 and 6.3.9 to obtain the following estimate,

$$\begin{aligned}\left|\int_0^t\int_0^L e\tau(S_E)_R dxds\right| &\le C_1\int_0^t(1+V_1(s))\left|\int_0^L\tau(S_E)_R dx\right|ds\\ &\le C_1\int_0^t(1+V_1(s))\left|\int_0^{+\infty}\int_{S^2} I\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right|ds\\ &\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\max\{\alpha-d,0\}}.\end{aligned}\tag{6.3.69}$$

Inserting (6.3.67)–(6.3.69) into (6.3.66), we get estimate (6.3.65). □

Now let us introduce

$$\begin{aligned}X(t) &:= \int_0^t\int_0^L(1+\theta)^{q+d}\theta_t^2\,dxds,\\ Y(t) &:= \int_0^L(1+\theta)^{2q}\theta_x^2\,dx,\\ K(\tau,\theta) &:= \int_0^\theta\frac{r^2\kappa(\tau,\xi)}{\tau}d\xi.\end{aligned}$$

Then we can easily derive from equation (6.2.9) that

$$\begin{cases}K_t = K_\tau(ru)_x + \dfrac{r^2\kappa(\tau,\theta)}{\tau}\theta_t,\\ K_{xt} = \left(\dfrac{r^2\kappa(\tau,\theta)}{\tau}\theta_x\right)_t + K_\tau(ru)_{xx} + K_{\tau\tau}(ru)_x\tau_x + \left(\dfrac{r^2\kappa(\tau,\theta)}{\tau}\right)_\tau\tau_x\theta_t.\end{cases}$$

Moreover, by assumptions (A_6) and (A_7), we get

$$|K_\tau| + |K_{\tau\tau}| \le C(1+\theta^{1+q}).\tag{6.3.70}$$

Lemma 6.3.11. *Under assumptions in theorem 6.2.1, the following estimates hold for any $t>0$,*

$$X(t)+Y(t)\le C_1,\tag{6.3.71}$$

$$\|\theta(s)\|_{L^\infty}\le C_1,\tag{6.3.72}$$

$$\begin{aligned}&\|\tau(t)-\overline{\tau}\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}(t)\right\|_{H^1}^2 + \left\|\theta(t)-\overline{\theta}\right\|_{H^1}^2\\ &\quad+\int_0^t\left(\|\tau-\overline{\tau}\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}\right\|_{H^2}^2 + \left\|\theta-\overline{\theta}\right\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}\right\|^2 + \|\theta_t\|^2\right)(s)\,ds \le C_1.\end{aligned}\tag{6.3.73}$$

Proof. Equation (6.2.13) can be rewritten as

$$e_\theta\theta_t = \left(\frac{\kappa r^2\theta_x}{\tau}\right)_x + \delta\frac{(ru)_x^2}{\tau} - \theta P_\theta(ru)_x + \mu\left(\frac{(rv)_x^2}{\tau} + \frac{r^2w_x^2}{\tau} - 2(u^2+v^2)_x\right) - \tau(S_E)_R. \tag{6.3.74}$$

Multiplying (6.3.74) by K_t over Q_t, we can obtain

$$\begin{aligned}&\int_0^t\int_0^L\left(e_\theta\theta_t + \theta P_\theta(ru)_x - \delta\frac{(ru)_x^2}{\tau} - \mu\frac{(rv)_x^2 + r^2w_x^2}{\tau}\right)K_t\,dxds\\&\quad + \int_0^t\int_0^L\frac{\kappa r^2\theta_x}{\tau}K_{tx}\,dxds + 2\mu\int_0^t\int_0^L(u^2+v^2)_xK_t\,dxds\\&\quad + \int_0^t\int_0^L\tau(S_E)_RK_t\,dxds = 0.\end{aligned} \tag{6.3.75}$$

Employing the technique of lemma 2.3.8 (see also lemma 2.1.9 in Qin [99]), similarly to the proof of lemma 2.8 in Qin *et al.* [106], we can estimate each term in (6.3.75). Obviously,

$$\int_0^t\int_0^L e_\theta\theta_t^2\frac{r^2\kappa}{\tau}\,dxds \geq C_1^{-1}X(t), \tag{6.3.76}$$

$$\left|\int_0^t\int_0^L e_\theta\theta_tK_\tau(ru)_x\,dxds\right| \leq \frac{1}{16}X(t) + C_1\left(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{1+\max\{\beta,d+2+\beta-q\}}. \tag{6.3.77}$$

Applying the Cauchy inequality and corollaries 6.3.1, 6.3.2 and lemma 6.3.8, we can derive

$$\begin{aligned}&\left|\int_0^t\int_0^L\left(\theta P_\theta(ru)_x - \delta\frac{(ru)_x^2}{\tau} - \mu\frac{(rv)_x^2 + r^2w_x^2}{\tau}\right)\frac{r^2\kappa}{\tau}\theta_t\,dxds\right|\\&\quad\leq \frac{1}{16}X(t) + C_1\int_0^t\int_0^L(1+\theta)^{q-d}|(ru)_x^4 + (rv)_x^4 + w_x^4|\,dxds\\&\qquad + C_1\left(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\right)^{1+\max\{\beta,d+2+\beta-q\}}.\end{aligned} \tag{6.3.78}$$

Using the Gagliardo–Nirenberg inequality and corollary 6.3.1 and lemma 6.3.8, we have

$$
\begin{aligned}
&\int_0^t\int_0^L(1+\theta)^{q-d}(ru)_x^4dxds\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_1}\int_0^t\int_0^L(1+\theta)^{\frac{q+d+1}{2}}(ru)_x^4dxds\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_1}\left(\int_0^t\big\|(ru)_x\big\|^3\big\|(ru)_{xx}\big\|ds+\int_0^t V_1^{\frac12}(s)\big\|(ru)_x\big\|^3\big\|(ru)_{xx}\big\|ds\right)\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_1}\left(\sup_{0\le s\le t}\big\|(ru)_x(s)\big\|^2\left(\int_0^t\big\|(ru)_x\big\|^2ds\right)^{\frac12}\left(\int_0^t\big\|(ru)_{xx}\big\|^2ds\right)^{\frac12}\right.\\
&\qquad\left.+\sup_{0\le s\le t}\big\|(ru)_x(s)\big\|^3\left(\int_0^t V_1(s)ds\right)^{\frac12}\left(\int_0^t\big\|(ru)_{xx}\big\|^2ds\right)^{\frac12}\right)\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_2}
\end{aligned}
\tag{6.3.79}
$$

with $q_1=\max\{\frac{q-3d-1}{2},0\}$, $q_2=q_1+2\max\{\beta,d+2+\beta-q\}$. Similarly, by virtue of lemma 6.3.7,

$$
\int_0^t\int_0^L(1+\theta)^{q-d}((rv)_x^4+w_x^4)dxds\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_1}.
\tag{6.3.80}
$$

Inserting (6.3.79)–(6.3.81) into (6.3.78), we have

$$
\begin{aligned}
&\left|\int_0^t\int_0^L\left(\theta P_\theta(ru)_x-\delta\frac{(ru)_x^2}{\tau}-\mu\frac{(rv)_x^2+r^2w_x^2}{\tau}\right)\frac{r^2\kappa}{\tau}\theta_tdxds\right|\\
&\quad\le\frac{1}{16}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_3}
\end{aligned}
\tag{6.3.81}
$$

with $q_3=\max\{1+\max\{\beta,d+2+\beta-q\},q_2\}$.

It follows from lemmas 6.3.8, 6.3.9 and corollaries 6.3.1, 6.3.2 that

$$
\begin{aligned}
&\left|\int_0^t\int_0^L\left(\theta P_\theta(ru)_x-\delta\frac{(ru)_x^2}{\tau}-\mu\frac{(rv)_x^2+r^2w_x^2}{\tau}\right)K_\tau(ru)_xdxds\right|\\
&\quad\le C_1\int_0^t\int_0^L\Big((1+\theta)^{q+d+2}|(ru)_x|^2+(1+\theta)^{q+1}|(ru)_x^3+(rv)_x^2(ru)_x+r^2w_x^2(ru)_x|\Big)dxds\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)\int_0^t\int_0^L(1+V_1(s))|(ru)_x|^2dxds\\
&\qquad+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\frac{q-d+1}{2}}\int_0^t\int_0^L(1+V_1^{\frac12}(s))\big\|(ru)_x\big\|_{L^\infty}((ru)_x^2+(rv)_x^2+r^2w_x^2)dxds\\
&\quad\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{1+\max\{\beta,d+2+\beta-q\}}+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_4}\\
&\qquad+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\frac{q-d+1}{2}+\max\{\beta,d+2+\beta-q\}}\left(\int_0^t V_1(s)ds\right)^{\frac12}\left(\int_0^t\big\|(ru)_x\big\|_{L^\infty}^2(s)ds\right)^{\frac12}
\end{aligned}
$$

$$\begin{aligned}
&\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{\frac{q-d+1}{2}+\max\{\beta,d+2+\beta-q\}}\bigg(\int_0^t\Big(\big\|(ru)_x\big\|^2+\big\|(ru)_{xx}\big\|^2\Big)(s)\,ds\bigg)^{\frac{1}{2}}\\
&\quad+C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{q_4}\\
&\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{q_4} \qquad (6.3.82)
\end{aligned}$$

with $q_4=\frac{q-d+1}{2}+2\max\{\beta,d+2+\beta-q\}$.

Now let us consider the various contributions in the second integral of (6.3.75). By lemmas 6.3.1–6.3.8 and 6.3.10 and corollaries 6.3.1, 6.3.2, we have

$$\int_0^t\int_0^L\frac{r^2\kappa\theta_x}{\tau}\Big(\frac{r^2\kappa\theta_x}{\tau}\Big)_t dxds\geq C_1^{-1}Y(t)-C_1, \qquad (6.3.83)$$

$$\begin{aligned}
\bigg|\int_0^t\int_0^L\frac{r^2\kappa\theta_x}{\tau}K_\tau(ru)_{xx}dxds\bigg|&\leq C_1\int_0^t\int_0^L(1+\theta)^{1+2q}|\theta_x(ru)_{xx}|dxds\\
&\leq C_1\bigg(\int_0^t\int_0^L(1+\theta)^{d+q}\theta_x^2dxds\bigg)^{\frac{1}{2}}\\
&\quad\times\bigg(\int_0^t\int_0^L(1+\theta)^{3q-d+2}|(ru)_{xx}|^2dxds\bigg)^{\frac{1}{2}}\\
&\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{q_5} \qquad (6.3.84)
\end{aligned}$$

and

$$\begin{aligned}
&\bigg|\int_0^t\int_0^L\frac{r^2\kappa\theta_x}{\tau}K_{\tau\tau}\tau_x(ru)_xdxds\bigg|\\
&\quad\leq C_1\int_0^t\int_0^L(1+\theta)^{1+2q}|\theta_x\tau_x(ru)_x|dxds\\
&\quad\leq C_1\bigg(\int_0^t\int_0^L(1+\theta)^{d+q}\theta_x^2dxds\bigg)^{\frac{1}{2}}\bigg(\int_0^t\int_0^L(1+\theta)^{3q-d+2}\tau_x^2|(ru)_x|^2dxds\bigg)^{\frac{1}{2}}\\
&\quad\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{\frac{3+2\beta+3q-d}{2}}\bigg(\int_0^t\big\|(ru)_x\big\|_{L^\infty}^2dxds\bigg)^{\frac{1}{2}}\\
&\quad\leq C_1\Big(1+\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^{q_5+\frac{\beta}{2}} \qquad (6.3.85)
\end{aligned}$$

with $q_5=\frac{1}{2}(3q-d+\beta+3+\max\{\beta,d+2+\beta-q\})$.

Using the embedding theorem and lemmas 6.3.7–6.3.9, we can easily derive

$$\int_0^t\int_0^L ((ru)_x^4+(rv)_x^4+w_x^4)\,dxds \le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{2\max\{\beta,d+2+\beta-q\}},$$

$$\begin{aligned}\int_0^t\int_0^L (u^2+v^2)_x^2\,dxds &\le C_1\int_0^t\Big(\|u_x\|_{L^\infty}^2+\|v_x\|_{L^\infty}^2\Big)(s)\,ds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\max\{\beta,d+2+\beta-q\}},\\ \int_0^t\int_0^L (\tau(S_E)_R)^2\,dxds &\le \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}(\tau^2\sigma_a^2I^2+\tau^2\sigma_a^2B^2+\tau^2\sigma_s^2(\widetilde{I}-I)^2)\,d\overrightarrow{\Omega}\,d\epsilon\,dxds\\ &\le C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\max\{\alpha-d,0\}}.\end{aligned}$$

Using the above facts and equation (6.2.13), we can obtain

$$\begin{aligned}\int_0^t\left\|\left(\frac{r^2\kappa\theta_x}{\tau}\right)_x\right\|^2 ds &\le C_1\int_0^t(\|e_\theta\theta_t\|^2+\|\theta P_\theta(ru)_x\|^2+\|(ru)_x^2\|^2+\|(rv)_x^2\|^2+\|w_x^2\|^2\\ &\quad+\|(u^2+v^2)_x\|^2+\|\tau(S_E)_R\|^2)\,ds\\ &\le C_1\int_0^t\int_0^L((1+\theta)^{2d}\theta_t^2+(1+\theta)^{2d+2}(ru)_x^2+(ru)_x^4+(rv)_x^4+w_x^4\\ &\quad+(u^2+v^2)_x^2)\,dxds+C_1\int_0^t\int_0^L(\tau(S_E)_R)^2\,dxds\\ &\le C_1\int_0^t\int_0^L(1+\theta)^{q+d}\theta_t^2\,dxds+C_1\int_0^t(1+V_1(s))\|(ru)_x\|^2\,ds\\ &\quad+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q_6}\\ &\le C_1\int_0^t\int_0^L(1+\theta)^{q+d}\theta_t^2\,dxds+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q_6}\end{aligned} \tag{6.3.86}$$

with $q_6=\max\{\alpha-d,2\max\{\beta,d+2+\beta-q\}\}$. Thus, by the Sobolev inequality, we can conclude

$$\begin{aligned}&\left|\int_0^t\int_0^L\frac{r^2\kappa\theta_x}{\tau}\left(\frac{r^2\kappa}{\tau}\right)_\tau\tau_x\theta_t\,dxds\right|\\ &\quad\le\frac{1}{16}X(t)+C_1\int_0^t\int_0^L(1+\theta)^{q-d}\left(\frac{r^2\kappa\theta_x}{\tau}\right)^2\tau_x^2\,dxds\\ &\quad\le\frac{1}{16}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q-d}\int_0^t\left\|\frac{r^2\kappa\theta_x}{\tau}\right\|_{L^\infty}^2\|\tau_x\|^2\,ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{16}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q-d+\beta}\int_0^t\Big\|\frac{r^2\kappa\theta_x}{\tau}\Big\|\Big\|\Big(\frac{r^2\kappa\theta_x}{\tau}\Big)_x\Big\|ds\\
&\le \frac{1}{16}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q-d+\beta}\Big(\int_0^t\Big\|\frac{r^2\kappa\theta_x}{\tau}\Big\|^2ds\Big)^{\frac12}\Big(\int_0^t\Big\|\Big(\frac{r^2\kappa\theta_x}{\tau}\Big)_x\Big\|^2ds\Big)^{\frac12}\\
&\le \frac{1}{16}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\frac32(q-d+\beta)+\frac12}\Big(\int_0^t\Big\|\Big(\frac{r^2\kappa\theta_x}{\tau}\Big)_x\Big\|^2ds\Big)^{\frac12}\\
&\le \frac{1}{8}X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q_7} \qquad (6.3.87)
\end{aligned}
$$

with $q_7=\frac32(q-d+\beta)+\frac12(1+q_6)$. By lemmas 6.3.7, 6.3.8 and corollary 6.3.1, using the Cauchy inequality, we have

$$
\begin{aligned}
&\Big|\int_0^t\int_0^L(u^2+v^2)_x\Big(K_\tau(ru)_x+\frac{r^2\kappa}{\tau}\theta_t\Big)dxds\Big|\\
&\quad\le \frac18 X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{\max\{\beta,d+2+\beta-q\}}. \qquad (6.3.88)
\end{aligned}
$$

The last contribution in (6.3.75) can be estimated as follows,

$$
\begin{aligned}
&\Big|\int_0^t\int_0^L\tau(S_E)_RK_tdxds\Big|\\
&\le\int_0^t\int_0^L\Big|\int_0^\infty\int_{S^2}\tau\sigma_aBd\overrightarrow{\mathbf{\Omega}}\,d\epsilon\Big||K_t|dxds\\
&\quad+\int_0^t\int_0^L\Big|\int_0^\infty\int_{S^2}\tau\sigma_aId\overrightarrow{\mathbf{\Omega}}\,d\epsilon\Big||K_t|dxd\tau\\
&\quad+\int_0^t\int_0^L\Big|\int_0^\infty\int_{S^2}\tau\sigma_s(\widetilde{I}-I)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\Big||K_t|dxd\tau\\
&=: M_1+M_2+M_3. \qquad (6.3.89)
\end{aligned}
$$

It follows from lemmas 6.3.8–6.3.10 and corollaries 6.3.1, 6.3.2 that

$$
\begin{aligned}
M_1&\le C_1\int_0^t\int_0^L(1+\theta^\alpha)|K_t|dxds\\
&\le C_1\int_0^t\int_0^L(1+\theta^{q+\alpha+1})|(ru)_x|dxds+C_1\int_0^t\int_0^L(1+\theta^{q+\alpha})|\theta_t|dxds\\
&\le\frac18X(t)+C_1\Big(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big)^{q_8} \qquad (6.3.90)
\end{aligned}
$$

with $q_8=q+\alpha-d+\frac12\max\{\beta,d+2+\beta-q\}$. Using the assumption (A_9), the Cauchy inequality and lemmas 6.3.8–6.3.10, we have

$$\begin{aligned}M_2 &\le C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\left|\tau\sigma_a I\left(K_\tau(ru)_x+\frac{r^2\kappa}{\tau}\theta_t\right)\right|d\overrightarrow{\Omega}\,d\epsilon dxds\\ &\le C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\tau\sigma_a I^2 d\overrightarrow{\Omega}\,d\epsilon dxds+C_1\int_0^t\int_0^L(1+\theta^{2q+2})|(ru)_x|^2dxds\\ &\quad+C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}(1+\theta^q)|I\theta_t|d\overrightarrow{\Omega}\,d\epsilon dxds\\ &\le\frac{1}{8}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\max\{q-d+\max\{\beta,d+2+\beta-q\},\alpha-d\}}.\end{aligned}\tag{6.3.91}$$

Using the same technique, we also get

$$M_3\le\frac{1}{8}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{\max\{q-d+\max\{\beta,d+2+\beta-q\},\alpha-d\}}.\tag{6.3.92}$$

Inserting the estimates (6.3.90)–(6.3.92) into (6.3.89), we get

$$\left|\int_0^t\int_0^L\tau(S_E)_R K_t dxd\tau\right|\le\frac{1}{8}X(t)+C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_9}\tag{6.3.93}$$

with $q_9=\max\bigl\{2+\beta,q-d+\beta,q+\alpha-d+\frac{1}{2}\beta,\frac{1}{2}q+\alpha-\frac{1}{2}d+1\bigr\}$.

Inserting all previous estimates (6.3.76), (6.3.77), (6.3.81)–(6.3.85), (6.3.87), (6.3.88) and (6.3.93) into (6.3.75), we obtain

$$X(t)+Y(t)\le C_1\left(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\right)^{q_{10}}\tag{6.3.94}$$

with $q_{10}=\max\bigl\{1+\max\{\beta,d+2+\beta-q\},q_4,q_5+\frac{1}{2}\beta,q_7,q_9\bigr\}$. Similarly to the proof of lemma 2.3.9, we have

$$\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\le C_1+C_1Y^{\frac{1}{2q+d+3}}(t).$$

By a direct calculation, we can know that $q_{10}<2q+d+3$. Thus by the Young inequality, it follows that

$$X(t)+Y(t)\le C_1$$

which yields

$$\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\le C_1.$$

Combining lemmas 6.3.6–6.3.10, we can obtain (6.3.73). Therefore, we complete the proof. □

The following two lemmas concern with the uniform estimate of radiative intensity.

Lemma 6.3.12. *The following estimates hold that for any* $t > 0$,

$$\left\|\int_0^{+\infty}\int_{S^2} I\, d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} + \left\|\int_0^{+\infty}\int_{S^2} \widetilde{I}\, d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} \le C_1, \tag{6.3.95}$$

$$\left\|\int_0^{+\infty}\int_{S^2} I^2\, d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} + \left\|\int_0^{+\infty}\int_{S^2} |I_x|\, d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} \le C_1, \tag{6.3.96}$$

$$\left\|\int_0^{+\infty}\int_{S^2} |I_t|\, d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} + \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_t^2(x, s; \epsilon, \overrightarrow{\Omega})\, d\overrightarrow{\Omega}\, d\epsilon dx ds \le C_1. \tag{6.3.97}$$

Proof. In view of lemma 6.3.9 and (6.3.73), we easily get (6.3.95). Applying the Young and the Hölder inequalities, we derive

$$\begin{aligned}\int_0^{+\infty}\int_{S^2} I^2\, d\overrightarrow{\Omega}\, d\epsilon &\le \int_0^{+\infty}\int_{S^2}\left(\int_0^L \frac{\tau}{r\omega}(\sigma_a B + \sigma_s \widetilde{I})\, dy\right)^2 d\overrightarrow{\Omega}\, d\epsilon\\ &\le C_1\int_0^{+\infty}\int_{S^2}\left(\int_0^L \frac{\sigma_a \tau^2}{r^2\omega^2} dx \cdot \int_0^L \sigma_a B^2\, dx\right) d\overrightarrow{\Omega}\, d\epsilon\\ &\quad + C_1\int_0^{+\infty}\int_{S^2}\left(\int_0^L \frac{\sigma_s \tau^2}{r^2\omega^2} dx \cdot \int_0^L \sigma_s \widetilde{I}^2\, dx\right) d\overrightarrow{\Omega}\, d\epsilon\\ &\le C_1 + C_1\int_0^{+\infty}\int_{S^2}\left(\int_0^L \sigma_s(\widetilde{I} - I)^2 + I^2\right) dx d\overrightarrow{\Omega}\, d\epsilon \le C_1\end{aligned}$$

which implies

$$\left\|\int_0^{+\infty}\int_{S^2} I^2 d\overrightarrow{\Omega}\, d\epsilon\right\|_{L^\infty(Q_t)} \le C_1. \tag{6.3.98}$$

It follows from equation (6.2.14) that

$$I_x = -\left(\frac{\tau}{r\omega}(\sigma_a + \sigma_s) + \frac{\tau}{r^2}\right) I + \frac{\tau}{r\omega}(\sigma_a B + \sigma_s \widetilde{I}).$$

Integrating the above equality and using assumptions (A_8)–(A_{11}), we can derive

$$\begin{aligned}\int_0^{+\infty}\int_{S^2} |I_x|\, d\overrightarrow{\Omega}\, d\epsilon &\le C_1\int_0^{+\infty}\int_{S^2} \frac{(\sigma_a + \sigma_s)|I|}{|\omega|} d\overrightarrow{\Omega}\, d\epsilon + C_1\int_0^{+\infty}\int_{S^2} |I|\, d\overrightarrow{\Omega}\, d\epsilon\\ &\quad + C_1\int_0^{+\infty}\int_{S^2} \frac{(\sigma_a|B| + \sigma_s|\widetilde{I}|)}{|\omega|} d\overrightarrow{\Omega}\, d\epsilon\\ &\le C_1 + C_1\int_0^{+\infty}\int_{S^2} |I|\, d\overrightarrow{\Omega}\, d\epsilon + C_1\int_0^{+\infty}\int_{S^2} |\widetilde{I}|\, d\overrightarrow{\Omega}\, d\epsilon\end{aligned}$$

which, along with (6.3.95) and (6.3.98), leads to

$$\left\| \int_0^{+\infty} \int_{S^2} |I_x| \, d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \right\|_{L^\infty(Q_t)} \leq C_1. \tag{6.3.99}$$

It follows from (6.3.63) that for any $\omega \in (0,\, 1)$,

$$\begin{aligned} I_t =& \frac{1}{r}\int_0^x \exp\left(\int_x^y \frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)\left(\int_x^y \frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)_t \frac{\tau}{\omega}(\sigma_a B+\sigma_s \widetilde{I})\,dy \\ &+ \frac{1}{r}\int_0^x \exp\left(\int_x^y \frac{\tau(\sigma_a+\sigma_s)}{r\omega}dz\right)\left(\frac{\tau}{\omega}(\sigma_a B+\sigma_s \widetilde{I})\right)_t dy - \frac{u}{r}I \\ =&: N_1 + N_2 - \frac{u}{r}I. \end{aligned} \tag{6.3.100}$$

Using the Young inequality, lemmas 6.3.1–6.3.11 and assumptions (A_9)–(A_{11}) and (A_{13}), we have

$$\begin{aligned} &\int_0^t \int_0^{+\infty} \int_{S^2} N_1^2 \, d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \left(\int_0^x \int_x^y \left(\frac{1}{\omega}((r^{-1}\tau)_t(\sigma_a+\sigma_s) + r^{-1}\tau(\sigma_a+\sigma_s)_\tau \tau_t \right.\right. \\ &\quad \left.\left. + r^{-1}\tau(\sigma_a+\sigma_s)_\theta \theta_t) \right) dz \cdot \frac{\tau(\sigma_a B + \sigma_s \widetilde{I})}{r^{n-1}\omega} \, dy \right)^2 d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \left(\int_0^L \left(\frac{(ru)_x^2}{\omega^2}(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_\tau)^2 \right.\right. \\ &\quad \left.\left. + \frac{\theta_t^2}{\omega^2}(\sigma_a+\sigma_s)_\theta^2 \right) dx \right) \int_0^L \frac{\tau^2}{\omega^2}(\sigma_a^2 B^2 + \sigma_s^2 \widetilde{I}^2) \, dx d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \int_0^t \int_0^L (\tau_t^2 + \theta_t^2) \, dx ds + C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \int_0^L \widetilde{I}^2 \, dx d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \end{aligned} \tag{6.3.101}$$

and

$$\begin{aligned} &\int_0^t \int_0^{+\infty} \int_{S^2} N_2^2 \, d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \left(\int_0^x \left(\frac{1}{\omega}(\tau_t(\sigma_a B + \sigma_s \widetilde{I}) + ((\sigma_a)_\tau \tau_t B \right.\right. \\ &\quad \left.\left. + (\sigma_a)_\theta \theta_t B + \sigma_a B_\theta \theta_t + (\sigma_s)_\tau \tau_t \widetilde{I} + (\sigma_s)_\theta \theta_t \widetilde{I} + \sigma_s \widetilde{I}_t \right) dy \right)^2 d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 \int_0^t \int_0^L (\tau_t^2 + \theta_t^2) \, dx d\tau + C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \int_0^x I_t^2 \, dy d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \\ &\leq C_1 + C_1 \int_0^t \int_0^{+\infty} \int_{S^2} \int_0^x I_t^2 \, dy d\overrightarrow{\mathbf{\Omega}} \, d\epsilon \, ds \end{aligned}$$

which, together with (6.3.98) and (6.3.101), implies

$$\int_0^t\int_0^{+\infty}\int_{S^2} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon ds \le C_1 + C_1\int_0^x\left(\int_0^t\int_0^{+\infty}\int_{S^2} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon ds\right)dy. \tag{6.3.102}$$

Using the same technique, we have the above inequality for any $\omega \in (-1,\, 0)$. By the Gronwall inequality, we have

$$\int_0^t\int_0^{+\infty}\int_{S^2} I_t^2\, d\overrightarrow{\Omega}\, d\epsilon ds \le C_1 e^{C_1 x} \le C_1 e^{C_1 L} \le C_1. \tag{6.3.103}$$

Similarly, we can obtain

$$\int_0^{+\infty}\int_{S^2} |I_t|\, d\overrightarrow{\Omega}\, d\epsilon \le C_1. \tag{6.3.104}$$

Thus, the estimate (6.3.97) follows from (6.3.103) and (6.3.104). □

Lemma 6.3.13. *Under assumptions in theorem 6.2.1, the following estimates hold that for any* $t > 0$,

$$\|\mathcal{I}_x(t)\| \le C_1, \qquad \|\mathcal{I}_{xx}(t)\| \le C_1. \tag{6.3.105}$$

Proof. Applying lemmas 6.3.9 and 6.3.12, we have

$$\begin{aligned}\|\mathcal{I}_x(t)\|^2 &\le \int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\left(\frac{\sigma_a+\sigma_s}{|\omega|}+\tau\right)|I| + \sigma_a B + \sigma_s|\widetilde{I}|\right)d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &\le C_1 + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}|I|\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}|\widetilde{I}|\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &\le C_1. \end{aligned}\tag{6.3.106}$$

By virtue of a direct computation, we also have

$$\begin{aligned}\|\mathcal{I}_{xx}(t)\|^2 &= \int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}\left((r^{-1}\tau)_x S + r^{-1}\tau S_x\right) - \left((r^{-2}\tau)_x I + r^{-2}\tau I_x\right)\right)d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &\le C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\frac{r^{-1}\tau}{\omega}S_x\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\frac{(r^{-1}\tau)_x}{\omega}S\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &\quad + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\tau_x I\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2} I_x\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &\quad + C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2} I\, d\overrightarrow{\Omega}\, d\epsilon\right)^2 dx\\ &=: G_1 + G_2 + G_3 + C_1\|\mathcal{I}_x\|^2 + C_1\|\mathcal{I}\|^2. \end{aligned}\tag{6.3.107}$$

Similarly to the proof of lemma 5.3.10 (see also lemma 2.10 in Qin *et al.* [106]), using lemma 6.3.12, we see that

$$G_1 \leq C_1 + C_1 \int_0^L (\tau_x^2 + \theta_x^2)\,dx \leq C_1, \tag{6.3.108}$$

$$G_2 \leq C_1 \int_0^L (r^{-1}\tau)_x^2\,dx \leq C_1. \tag{6.3.109}$$

Using (6.3.55) and (6.3.95), we can obtain

$$G_3 = C_1 \int_0^L \left(\int_0^{+\infty} \int_{S^2} \tau_x I d\overrightarrow{\Omega}\, d\epsilon \right)^2 dx \leq C_1 \int_0^L \tau_x^2\,dx \left\| \int_0^{+\infty} \int_{S^2} I\, d\overrightarrow{\Omega}\, d\epsilon \right\|_{L^\infty}^2 \leq C_1. \tag{6.3.110}$$

Inserting (6.3.108)–(6.3.110) into (6.3.107), we obtain the desired estimate. □

Finally, we concern estimate on the first-order derivative of θ.

Lemma 6.3.14. *Under assumptions in theorem 6.2.1, the following estimate holds for any* $t > 0$,

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(s)\|^2\,ds \leq C_1. \tag{6.3.111}$$

Proof. Multiplying (6.3.74) by $e_\theta^{-1}\theta_{xx}$, integrating the resultant over $(0,\ L)$ and using the Young inequality, the interpolation inequality and lemmas 6.3.1–6.3.5, 6.3.7 and 6.3.8, we can conclude for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\|\theta_x\|^2 + 2\int_0^L \frac{r^2\kappa}{e_\theta \tau}\theta_{xx}^2\,dx \\
&\quad = 2\int_0^L e_\theta^{-1}\left(\theta P_\theta (ru)_x - \delta\frac{(ru)_x^2}{\tau} - \left(\frac{r^2\kappa}{\tau}\right)_x \theta_x \right.\\
&\qquad \left. - \mu\frac{(rv)_x^2 + r^2 w_x^2}{\tau} + 2\mu(u^2+v^2)_x + \tau (S_E)_R\right)\theta_{xx}\,dx \\
&\quad \leq \frac{\varepsilon}{2}\|\theta_{xx}\|^2 + C_1\big(\left\|(ru)_x\right\|^2 + \left\|(ru)_x\right\|_{L^4}^2 + \|\tau_x\theta_x\|^2 \\
&\qquad + \|\theta_x\|^2 + \|\theta_x\|_{L^4}^2 + \left\|(rv)_x\right\|_{L^4}^2 + \|w_x\|_{L^4}^2 + \left\|(S_E)_R\right\|^2\big) \\
&\quad \leq \varepsilon\|\theta_{xx}\|^2 + C_1\Big(\left\|\overrightarrow{\mathbf{v}}_x\right\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xx}\right\|^2 + \|\theta_x\|^2 + \left\|(S_E)_R\right\|^2\Big).
\end{aligned} \tag{6.3.112}$$

Now we need to estimate the term $\left\|(S_E)_R\right\|^2$ in (6.3.112). By virtue of assumptions (A_8)–(A_{11}), we derive from (6.3.58), (6.3.96) and the Young and the Hölder inequalities that

$$\begin{aligned}\left\|(S_E)_R\right\|^2 &\leq C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\sigma_a(B-I)\,d\overrightarrow{\Omega}\,d\epsilon\right)^2dx\\&\quad+C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\sigma_s(\widetilde{I}-I)\,d\overrightarrow{\Omega}\,d\epsilon\right)^2dx\\&\leq C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\sigma_a\,d\overrightarrow{\Omega}\,d\epsilon\int_0^{+\infty}\int_{S^2}\sigma_a(B^2+I^2)\,d\overrightarrow{\Omega}\,d\epsilon\right)dx\\&\quad+C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\sigma_s\,d\overrightarrow{\Omega}\,d\epsilon\int_0^{+\infty}\int_{S^2}\sigma_s(\widetilde{I}-I)^2\,d\overrightarrow{\Omega}\,d\epsilon\right)dx\\&\leq C_1\int_0^L\theta^{1+\alpha}dx+C_1\leq C_1 \qquad (6.3.113)\end{aligned}$$

which, together with (6.3.112), leads to (6.3.111). □

6.3.2 Asymptotic Behavior in $\mathcal{H}^1$

This subsection is aimed at showing the asymptotic behavior of solutions in $\mathcal{H}^1$ to the problem (6.2.9)–(6.2.17) by the following two lemmas.

Lemma 6.3.15. *Under assumptions in theorem 6.2.1, we have, as $t\to+\infty$,*

$$\|\tau(t)-\overline{\tau}\|_{H^1}\to 0, \qquad (6.3.114)$$

$$\left\|\overrightarrow{\mathbf{v}}(t)\right\|_{H^1}\to 0, \qquad (6.3.115)$$

$$\left\|\theta(t)-\overline{\theta}\right\|_{H^1}\to 0, \quad \left\|\theta(t)-\overline{\theta}\right\|_{L^\infty}\to 0 \qquad (6.3.116)$$

where constant $\overline{\theta}>0$ is determined by $e(\overline{\tau},\overline{\theta})=\int_0^L\left(\frac{1}{2}|\overrightarrow{\mathbf{v}}_0|^2+e(\tau_0,\theta_0)\right)dx$ and for all $(x,\,t)\in[0,\,L]\times[0,\,+\infty)$,

$$0<C_1^{-1}\leq\theta(x,t)\leq C_1. \qquad (6.3.117)$$

Proof. Similarly to the proof of lemma 5.3.11, we can prove this lemma. Thus we omit the detail. □

Lemma 6.3.16. *Under assumptions in theorem 6.2.1, we have as, $t\to+\infty$,*

$$\|\mathcal{I}(t)\|_{H^2}\to 0. \qquad (6.3.118)$$

Proof. Similarly to the proof of lemma 5.3.12, we can also obtain (6.3.118). Here we omit it. □

Till now we have completed the proof of theorem 6.2.1. □

6.4 Global Existence and Asymptotic Behavior in $\mathcal{H}^2$

In this section, we shall show theorem 6.2.2, that is, the global existence and asymptotic behavior of solutions in $\mathcal{H}^2$ to the problem (6.2.9)–(6.2.17) under some relative assumptions.

6.4.1 Global Existence in $\mathcal{H}^2$

We first begin with some estimates on the second-order derivative of velocity and temperature.

Lemma 6.4.1. *Under assumptions of theorem 6.2.2, there holds that for any* $t > 0$,

$$\|\vec{\mathbf{v}}_t(t)\|^2 + \|\vec{\mathbf{v}}_{xx}(t)\|^2 + \|\vec{\mathbf{v}}_x(t)\|^2_{L^\infty} + \int_0^t \left(\|\vec{\mathbf{v}}_{xt}\|^2 + \|\vec{\mathbf{v}}_{xx}\|^2\right)(s)\,ds \le C_2, \tag{6.4.1}$$

$$\|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|\theta_x(t)\|^2_{L^\infty} + \int_0^t \left(\|\theta_{xt}\|^2 + \|\theta_{xx}\|^2\right)(s)\,ds \le C_2. \tag{6.4.2}$$

Proof. Differentiating (6.2.10) with respect to t and then multiplying the result by u_t over Q_t, by theorem 6.2.1 and the Young inequality, we can derive

$$\begin{aligned}
&\|u_t(t)\|^2 + \int_0^t \|u_{xt}(s)\|^2 ds \\
&\quad\le C_2 + C_1 \int_0^t \int_0^L |(ru)_x \tau_t + P_\tau \tau_t + P_\theta \theta_t||u_{xt}|\,dxds \\
&\qquad + C_1 \int_0^t \int_0^L \left| u_t \left(\delta \frac{(ru)_x}{\tau} - P\right)_x \right| dxds + C_1 \int_0^t \int_0^L |u_t(vv_t - v^2 u)\,dxds \\
&\quad\le C_2 + \varepsilon \int_0^t \int_0^L u_{xt}^2\,dxds + C_2(\varepsilon) \int_0^t \|(ru)_x\|^2_{L^\infty} \int_0^L (ru)_x^2\,dxds \\
&\qquad + C_2 \int_0^t \|u_x\|^2_{L^\infty} \int_0^L \tau_x^2\,dxds \\
&\quad\le C_2 + \varepsilon \int_0^t \int_0^L u_{xt}^2\,dxds.
\end{aligned} \tag{6.4.3}$$

Taking $\varepsilon > 0$ small enough, we have

$$\|u_t(t)\|^2 + \int_0^t \|u_{xt}(s)\|^2 ds \le C_2. \tag{6.4.4}$$

Noting from (6.2.10) that

$$\|u_{xx}\| \le C_2\left(\|u_t\| + \|\tau_x\| + \|\theta_x\| + \|u_x\| + \|v\|^2\right) \tag{6.4.5}$$

and by the embedding theorem, we have

$$\|u_{xx}(t)\|^2+\|u_t(t)\|^2+\|u_x(t)\|_{L^\infty}^2+\int_0^t\|u_{xt}(s)\|^2ds\leq C_2. \tag{6.4.6}$$

Similarly, we can derive the similar estimates on v, w from (6.2.11), (6.2.12) and then obtain estimate (6.4.1).

Analogously, using the method similar to that in lemma 5.4.1, differentiating (6.2.13) with respect to t, then multiplying the result by θ_t over Q_t, and using the Young inequality and the embedding theorem, we can derive,

$$\|\theta_t(t)\|^2+\int_0^t\|\theta_{xt}(\tau)\|^2ds\leq C_2+C_1\sup_{0\leq s\leq t}\|\theta_t(s)\|^{\frac{4}{3}}+C_1\int_0^t\left\|(\tau(S_E)_R)_t\right\|^2ds. \tag{6.4.7}$$

It follows from lemmas 6.3.9–6.3.12 that

$$\begin{aligned}
&\int_0^t\left\|(\tau(S_E)_R)_t\right\|^2ds\\
&=\int_0^t\int_0^L\left((ru)_x(S_E)_R+\tau((S_E)_R)_t\right)^2dxds\\
&\leq C_1\int_0^t\int_0^L(ru)_x^2\left(\int_0^{+\infty}\int_{S^2}(\sigma_a(B-I)+\sigma_s(\widetilde{I}-I))d\overrightarrow{\mathbf{\Omega}}d\epsilon\right)^2dxds\\
&\quad+C_1\int_0^t\int_0^L\left(\int_0^{+\infty}\int_{S^2}(((\sigma_a)_\tau(ru)_x+(\sigma_a)_\theta\theta_t)(B-I)+\sigma_a(B_\theta\theta_t-I_t)\right.\\
&\quad\left.+((\sigma_s)_\tau(ru)_x+(\sigma_s)_\theta\theta_t)(\widetilde{I}-I)+\sigma_s(\widetilde{I}-I)_t)d\overrightarrow{\mathbf{\Omega}}d\epsilon\right)^2dxds\\
&\leq C_1\int_0^t\left(\left\|(ru)_x\right\|^2+\|\theta_t\|^2\right)ds+C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}I_t^2d\overrightarrow{\mathbf{\Omega}}d\epsilon dxds\\
&\leq C_1.
\end{aligned} \tag{6.4.8}$$

Inserting (6.4.8) into (6.4.7), we obtain

$$\|\theta_t(t)\|^2+\int_0^t\|\theta_{xt}(s)\|^2ds\leq C_2. \tag{6.4.9}$$

Using the Gagliardo–Nirenberg interpolation inequality and the Young inequality, we derive from (6.2.13) that

$$\|\theta_{xx}\|\leq C_1\left(\|\theta_t\|+\|\theta_x\|+\left\|(r\overrightarrow{\mathbf{v}})_x\right\|+\left\|\overrightarrow{\mathbf{v}}_x\right\|+\|\tau_x\|+\left\|\tau(S_E)_R\right\|\right)\leq C_2. \tag{6.4.10}$$

Combining (6.4.9), (6.4.10) and lemma 6.3.14, we get (6.4.2). □

The next lemma concerns estimate on the second-order derivative of the specific volume.

Lemma 6.4.2. *Under assumptions of theorem 6.2.2, the following estimates hold that for any* $t>0$,

$$\|\tau_{xx}(t)\|^2 + \int_0^t \|\tau_{xx}(s)\|^2 ds \le C_2, \tag{6.4.11}$$

$$\int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{xxx}\right\|^2 + \|\theta_{xxx}\|^2\right)(s)\, ds \le C_2. \tag{6.4.12}$$

Proof. Similarly to the proof of lemma 2.4.2, we have the following estimate

$$\|\tau_{xx}(t)\|^2 + \int_0^t \left(\|\tau_{xx}\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xxx}\right\|^2\right)(s)\, ds \le C_2. \tag{6.4.13}$$

Differentiating (6.2.13) with respect to x, using the Young inequality, the Gagliardo–Nirenberg interpolation and the Poincaré inequality and lemmas 6.3.1–6.3.14, we derive

$$\|\theta_{xxx}\| \le C_2\Big(\|\theta_{xt}\| + \|\theta_x\|_{H^1} + \left\|\overrightarrow{\mathbf{v}}_x\right\|_{H^1} + \left\|(\tau(S_E)_R)_x\right\|\Big). \tag{6.4.14}$$

By the definition of $(S_E)_R$ and lemma 6.3.12, it follows from that

$$\begin{aligned}\int_0^t \left\|\left(\tau(S_E)_R\right)_x\right\|^2 ds &= \int_0^t \int_0^L \left(\tau_x(S_E)_R + \tau\left((S_E)_R\right)_x\right)^2 dxds\\ &\le C_1 \int_0^t \left(\|\tau_x\|^2 + \|\theta_x\|^2\right) ds + C_1 \left\|\int_0^{+\infty}\int_{S^2} |I_x|\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right\|^2_{L^\infty(Q_t)}\\ &\le C_1 \end{aligned} \tag{6.4.15}$$

which, together with (6.4.13) and lemma 6.4.1, implies (6.4.12). ☐

Now we deal with the radiative intensity $\mathcal{I}$ in $\mathcal{H}^2$.

Lemma 6.4.3. *Under assumptions of theorem 6.2.2, the following estimate holds for any* $t > 0$,

$$\|\mathcal{I}_{xxx}(t)\| \le C_2. \tag{6.4.16}$$

Proof. It follows from (6.2.14) and the definitions of I and $\mathcal{I}$ that

$$\begin{aligned}\|\mathcal{I}_{xxx}\|^2 &= \int_0^L \left(\int_0^{+\infty}\int_{S^2} \left(\frac{1}{\omega}(r^{-1}\tau S)_{xx} - (r^{-2}\tau I)_{xx}\right) d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx\\ &\le C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2} \frac{(r^{-1}\tau)_{xx}}{\omega} S d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx + C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2} \frac{(r^{-1}\tau)_x}{\omega} S_x d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx\\ &\quad + C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2} \frac{r^{-1}\tau}{\omega} S_{xx} d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx + C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2} (r^{-2}\tau I)_{xx} d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx\\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{6.4.17}$$

Using the same technique as in lemma 5.4.3, we derive from lemmas 6.3.12, 6.3.13 and theorem 6.2.1 that with small mollification (here we omit the detail)

$$J_1 + J_2 + J_3 \le C_2. \tag{6.4.18}$$

Employing the Gagliardo–Nirenberg interpolation inequality and using lemmas 6.3.12 and 6.3.13, we can conclude

$$\begin{aligned} J_4 \le{}& C_1 \int_0^L (r^{-2}\tau)_{xx}^2 \left(\int_0^{+\infty}\int_{S^2} I\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx \\ &+ C_1 \int_0^L (r^{-2}\tau)_x^2 \left(\int_0^{+\infty}\int_{S^2} I_x\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx \\ &+ C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2} I_{xx}\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)^2 dx \\ \le{}& C_1 + C_1\left(\|\tau_x\|_{H^1}^2 + \|\mathcal{I}_{xx}\|^2\right) \le C_2 \end{aligned} \tag{6.4.19}$$

which, along with (6.4.17) and (6.4.18), leads to (6.4.16). □

6.4.2 Asymptotic Behavior in $\mathcal{H}^2$

Now we concern the asymptotic behavior of the global solutions in $\mathcal{H}^2$ by the next lemma.

Lemma 6.4.4. *Under assumptions in theorem 6.2.2, we have, as* $t \longrightarrow +\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^2}\to 0, \quad \left\|\overrightarrow{\mathbf{v}}(t)\right\|_{H^2}\to 0, \quad \left\|\theta(t)-\overline{\theta}\right\|_{H^2}\to 0, \quad \|\mathcal{I}(t)\|_{H^3}\to 0. \tag{6.4.20}$$

Proof. By a method similar to that in lemma 5.4.4, we have

$$\|\tau(t)-\overline{\tau}\|_{H^2}\to 0, \quad \left\|\overrightarrow{\mathbf{v}}(t)\right\|_{H^2}\to 0. \tag{6.4.21}$$

Similarly, we can also derive from (6.2.13) that

$$\frac{d}{dt}\|\theta_t\|^2 + C_1^{-1}\|\theta_{xt}\|^2 \le C_2\left(\|\theta_t\|^2 + \left\|\overrightarrow{\mathbf{v}}_x\right\|^2 + \left\|\overrightarrow{\mathbf{v}}_t\right\|^2 + \|\theta_x\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xt}\right\|^2 + \left\|\left(\tau (S_E)_R\right)_t\right\|^2\right)$$

which, along with theorem 1.2.10, lemma 6.4.1 and (6.4.8), gives, as $t \longrightarrow +\infty$,

$$\|\theta_t(t)\| \to 0. \tag{6.4.22}$$

We can derive from lemma 6.3.12 that

$$\begin{aligned} \frac{d}{dt}\left\|(S_E)_R\right\|^2 &= 2\int_0^L \left(\int_0^{+\infty}\int_{S^2} S\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right)\left(\int_0^{+\infty}\int_{S^2} S_t\, d\overrightarrow{\mathbf{\Omega}}\, d\epsilon\right) dx \\ &\le C_2\left(1 + \|u_x\|^2 + \|\theta_t\|^2\right) \end{aligned}$$

which, together with theorems 6.2.1 and 1.2.10, implies, as $t \to +\infty$,

$$\left\|(S_E)_R\right\| \to 0. \tag{6.4.23}$$

By equation (6.2.13), we see that

$$\|\theta_{xx}\| \le C_2\left(\|\theta_t\| + \|\theta_x\| + \|\tau_x\| + \left\|\overrightarrow{\mathbf{v}}_x\right\| + \left\|(S_E)_R\right\|\right), \tag{6.4.24}$$

which, along with lemma 6.3.15 and (6.4.22), (6.4.23), gives, as $t \to +\infty$,

$$\|\theta_{xx}(t)\| \to 0. \tag{6.4.25}$$

Thus we have, as $t \to +\infty$,

$$\left\|\theta(t) - \overline{\theta}\right\|_{H^2} \to 0.$$

Similarly to the proof of (5.4.21) in lemma 5.4.4, we can obtain the desired result $\|\mathcal{I}(t)\|_{H^3} \to 0$. □

Till now we have completed the proof of theorem 6.2.2. □

6.5 Global Existence and Asymptotic Behavior in $\mathcal{H}^4$

In this section, we shall prove theorem 6.2.3, that is, the global existence and asymptotic behavior of solutions in $\mathcal{H}^4$ to the problem (6.2.9)–(6.2.17) under some relative assumptions.

6.5.1 Global Existence in $\mathcal{H}^4$

First, we estimate u_{tt}, v_{tt}, w_{tt} and θ_{tt}.

Lemma 6.5.1. *Under assumptions of theorem 6.2.3, we have, for any $t > 0$ and $\varepsilon > 0$ small enough,*

$$\left\|\overrightarrow{\mathbf{v}}_{xt}(x,0)\right\| + \|\theta_{xt}(x,0)\| \le C_4, \tag{6.5.1}$$

$$\left\|\overrightarrow{\mathbf{v}}_{tt}(x,0)\right\| + \|\theta_{tt}(x,0)\| + \left\|\overrightarrow{\mathbf{v}}_{txx}(x,0)\right\| + \|\theta_{txx}(x,0)\| \le C_4, \tag{6.5.2}$$

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|\theta_{txx}\|^2 + \|u_{txx}\|^2 + \|v_{txx}\|^2\right)(s)\, ds, \tag{6.5.3}$$

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|u_{txx}\|^2 + \|v_{txx}\|^2\right)(s)\, ds, \tag{6.5.4}$$

$$\|w_{tt}(t)\|^2 + \int_0^t \|w_{ttx}(s)\|^2 ds \le C_4 + C_2 \int_0^t \left(\|u_{txx}\|^2 + \|w_{txx}\|^2\right)(s)\, ds, \tag{6.5.5}$$

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{ttx}(s)\|^2 ds \le C_4\varepsilon^{-3} + C_2\varepsilon^{-1} \int_0^t \|\theta_{txx}(s)\|^2 ds \\ + C_1\varepsilon \int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{ttx}\right\|^2 + \|u_{txx}\|^2\right)(s)\, ds.\end{aligned} \tag{6.5.6}$$

Proof. Similarly to the proof of lemma 5.5.1, we have the following estimate

$$\left\|\overrightarrow{\mathbf{v}}_{xt}(x,0)\right\| + \left\|\overrightarrow{\mathbf{v}}_{tt}(x,0)\right\| + \left\|\overrightarrow{\mathbf{v}}_{txx}(x,0)\right\| \le C_4. \tag{6.5.7}$$

Differentiating (6.2.13) with respect to x, using the Gagaliardo–Nirenberg inequality and the Young inequality, we can derive from theorem 6.2.1 and lemmas 6.4.1 and 6.4.2 that

$$\|\theta_{xt}(t)\| \le C_2\Big(\|\theta_x(t)\|_{H^2} + \|\tau_x(t)\|_{H^1} + \left\|\overrightarrow{\mathbf{v}}_x(t)\right\|_{H^1} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|\Big). \tag{6.5.8}$$

Similarly, differentiating (6.2.13) with respect to x twice, we can also derive

$$\|\theta_{xxt}(t)\| \le C_2\Big(\|\theta_x(t)\|_{H^3} + \|\tau_x(t)\|_{H^2} + \left\|\overrightarrow{\mathbf{v}}_x(t)\right\|_{H^2} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1}\Big) \tag{6.5.9}$$

or

$$\|\theta_{xxxx}(t)\| \le C_2\Big(\|\theta_x(t)\|_{H^2} + \|\tau_x(t)\|_{H^2} + \left\|\overrightarrow{\mathbf{v}}_x(t)\right\|_{H^2} + \|\theta_{txx}(t)\| + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1}\Big). \tag{6.5.10}$$

Differentiating (6.2.13) with respect to t, we have

$$\begin{aligned}\|\theta_{tt}(t)\| \le\ & C_2(\|\theta_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \left\|\overrightarrow{\mathbf{v}}_{tx}(t)\right\| + \|\tau_x(t)\| \\ & + \|\theta_x(t)\|\|\mathcal{I}_t(t)\|)\end{aligned} \tag{6.5.11}$$

$$\begin{aligned}\le\ & C_2(\left\|\overrightarrow{\mathbf{v}}_x(t)\right\|_{H^2} + \|\tau_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \|\theta_x(t)\|\|\mathcal{I}_x(t)\|_{H^1} \\ & + \|\theta_x(t)\|\|\mathcal{I}_t(t)\|).\end{aligned} \tag{6.5.12}$$

Thus estimates (6.5.1) and (6.5.2) follow from (6.5.7)–(6.5.12).

Similarly to lemma 2.5.1 (see also lemma 5.4.1 in Qin [99] or lemma 3.2 in Qin and Jiang [114]), we can obtain estimates (6.5.3)–(6.5.5).

Differentiating (6.2.13) with respect to t twice, multiplying the resultant by θ_{tt} and performing an integration by parts over $L^2(0,\ L)$, and using the embedding theorem and the Young inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tt}^2dx =& -\int_0^L\left(\frac{r^2\kappa\theta_x}{\tau}\right)_{tt}\theta_{ttx}dx-\int_0^L(e_{\theta tt}\theta_t+e_{\tau tt}(ru)_x)\theta_{tt}dx-\frac{3}{2}\int_0^L e_{\theta t}\theta_{tt}^2dx\\ &-\int_0^L\left(e_\tau+P-\delta\frac{(ru)_x}{\tau}\right)(ru)_{xtt}\theta_{tt}dx+\int_0^L\left(\delta\frac{(ru)_x}{\tau}-P\right)_{tt}(ru)_x\theta_{tt}dx\\ &-2\int_0^L\left(e_{\tau t}+\left(P-\delta\frac{(ru)_x}{\tau}\right)_t\right)(ru)_{xt}\theta_{tt}dx+\mu\int_0^L\left(\frac{(rv)_x^2+r^2w_x^2}{\tau}\right)_{tt}\theta_{tt}dx\\ &-2\mu\int_0^L(u^2+v^2)_{xtt}\theta_{tt}dx-\int_0^L\big(\tau(S_E)_R\big)_{tt}\theta_{tt}dx\\ =:&\sum_{i=1}^9A_i. \end{aligned}\tag{6.5.13}$$

Similarly to the proof of lemma 2.5.1, by virtue of theorems 6.2.1, 6.2.2 and lemmas 6.4.1–6.4.3, and using the embedding theorem and the Poincaré inequality, we can also derive that for any $\varepsilon\in(0,\,1)$,

$$A_1\le-(2C_1)^{-1}\|\theta_{ttx}\|^2+C_2\Big(\|\theta_{xt}\|^2+\|u_{xt}\|^2+\|u_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\theta_{tt}\|^2\Big),\tag{6.5.14}$$

$$A_2\le\varepsilon\|\theta_{ttx}\|^2+C_2\varepsilon^{-1}\Big(\|u_x\|_{H^1}^2+\|\theta_t\|^2+\|\theta_{xt}\|^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2\Big),\tag{6.5.15}$$

$$A_3\le C_1(\|\theta_{tt}\|+\|\theta_{ttx}\|)(\|u_x\|+\|\theta_t\|)\|\theta_{tt}\|\le\varepsilon\|\theta_{ttx}\|^2+C_2\varepsilon^{-1}\|\theta_{tt}\|^2,\tag{6.5.16}$$

$$A_4\le\varepsilon\|u_{ttx}\|^2+C_2\varepsilon^{-1}\|\theta_{tt}\|^2,\tag{6.5.17}$$

$$A_5\le\varepsilon\|u_{ttx}\|^2+C_2\varepsilon^{-1}\Big(\|\theta_{tt}\|^2+\|u_x\|_{H^1}^2+\|\theta_t\|^2+\|\theta_{xt}\|^2+\|u_{xt}\|^2\Big),\tag{6.5.18}$$

$$A_6\le C_2\|u_{tx}\|^{\frac{1}{2}}\|u_{txx}\|^{\frac{1}{2}}(\|u_x\|+\|\theta_t\|+\|u_{xt}\|)\|\theta_{tt}\|,\tag{6.5.19}$$

$$A_7\le\varepsilon\Big(\|v_{ttx}\|^2+\|w_{ttx}\|^2\Big)^2+C_2\varepsilon^{-1}\Big(\|\overrightarrow{\mathbf{v}}_x\|_{H^1}^2+\|u_{xt}\|^2+\|\theta_{tt}\|^2\Big),\tag{6.5.20}$$

$$A_8\le\varepsilon\Big(\|u_{ttx}\|^2+\|v_{ttx}\|^2\Big)+C_2\varepsilon^{-1}\Big(\|u_x\|_{H^1}^2+\|v_x\|_{H^1}^2+\|u_{xt}\|^2+\|v_{xt}\|^2+\|\theta_{tt}\|^2\Big).\tag{6.5.21}$$

It follows from (6.5.19) by the Hölder inequality that

$$\begin{aligned}\int_0^t A_6\,d\tau &\leq C_2 \sup_{0\leq s\leq t}\|\theta_{tt}(s)\|\left(\int_0^t \|u_{txx}(s)\|^2\,ds\right)^{\frac14}\left(\int_0^t \|u_{tx}(s)\|^2\,ds\right)^{\frac14}\\ &\quad\times\left(\int_0^t \left(\|u_x\|^2+\|\theta_t\|^2+\|u_{tx}\|^2\right)(s)\,ds\right)^{\frac12}\\ &\leq \varepsilon\left(\sup_{0\leq s\leq t}\|\theta_{tt}(s)\|^2+\int_0^t \|u_{txx}(s)\|^2\,ds\right)+C_2\varepsilon^{-3}. \end{aligned}\tag{6.5.22}$$

Now we can estimate A_9 as

$$\begin{aligned}A_9 &\leq C_1\int_0^L (ru)_{xt}^2\,dx + C_1\|(ru)_x\|_{L^\infty}^2\int_0^L ((S_E)_R)_t^2\,dx + C_1\int_0^L \theta_{tt}^2\,dx + C_1\int_0^L ((S_E)_R)_{tt}^2\,dx\\ &\leq C_1\left(\|u_{xt}\|^2+\|\theta_{tt}\|^2+\|((S_E)_R)_t\|^2\right)+C_1\int_0^L ((S_E)_R)_{tt}^2\,dx. \end{aligned}\tag{6.5.23}$$

By the induction of (6.4.8), we can easily obtain

$$\|((S_E)_R)_t\|^2 \leq C_1\left(\|u_x\|^2+\|\theta_t\|^2+\|\mathcal{I}_t\|^2\right).\tag{6.5.24}$$

Using the Hölder inequality and the interpolation theorem, we can derive from theorems 6.2.1 and 6.2.2 that

$$\begin{aligned}\int_0^L ((S_E)_R)_{tt}^2\,dx &= \int_0^L \left(\int_0^{+\infty}\int_{S^2}(\sigma_a(B-I)+\sigma_s(\mathcal{I}-I)\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)_{tt}^2 dx\\ &\leq C_1\int_0^L ((ru)_x^4+(ru)_x^2\theta_t^2+\theta_t^4+\theta_{tt}^2+(ru)_{xt}^2)\,dx + C_1\int_0^L\int_0^{+\infty}\int_{S^2} I_{tt}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\ &\leq C_1\left(\|u_x\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\|\theta_{tt}\|^2+\|u_{xt}\|^2\right)+C_1\int_0^L\int_0^{+\infty}\int_{S^2} I_{tt}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx. \end{aligned}\tag{6.5.25}$$

Differentiating (6.3.67) with respect to t twice and after the lengthy calculation, we can derive

$$\int_0^t\int_0^{+\infty}\int_{S^2} I_{tt}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,ds \leq C_2 + C_2\int_0^t \|\theta_{tt}(s)\|^2\,ds + C_2\int_0^x\int_0^t\int_0^{+\infty}\int_{S^2} I_{tt}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,ds\,dy.\tag{6.5.26}$$

Applying the Gronwall inequality to (6.5.26), we can obtain

$$\begin{aligned}\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_{tt}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds &\leq \left(C_2+C_2\int_0^t \|\theta_{tt}(s)\|^2\,ds\right)e^{C_2x}\\ &\leq C_2+C_2\int_0^t \|\theta_{tt}(s)\|^2\,ds. \end{aligned}\tag{6.5.27}$$

Inserting (6.5.24), (6.5.25) and (6.5.27) into (6.5.23), we have

$$A_9 \leq C_2\Big(\|u_{xt}\|^2 + \|\theta_{tt}\|^2 + \|u_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\mathcal{I}_t\|^2\Big). \tag{6.5.28}$$

Thus we derive from (6.5.13)–(6.5.18), (6.5.20), (6.5.21) and (6.5.28) that for any $\varepsilon \in (0, 1)$ small enough,

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{ttx}(s)\|^2 ds \leq C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t \|\theta_{tt}(s)\|^2 ds + C_1\varepsilon\Big(\sup_{0\leq s\leq t}\|\theta_{tt}(s)\|^2 \\ + \int_0^t \Big(\|u_{txx}\|^2 + \|\overrightarrow{\mathbf{v}}_{ttx}\|^2\Big)(s)\, ds\Big).\end{aligned} \tag{6.5.29}$$

Therefore taking supremum in t on the left-hand side of (6.5.29), picking $\varepsilon \in (0, 1)$ small enough, and using (6.5.11), we can derive estimate (6.5.6). The proof is thus complete. □

Next, similarly to lemma 2.5.2, we can establish estimates on u_{xt}, v_{xt}, w_{xt} and θ_{xt}.

Lemma 6.5.2. *Under assumptions of theorem 6.2.3, the following estimates hold that for any $t > 0$ and $\varepsilon > 0$ small enough,*

$$\|u_{xt}(t)\|^2 + \int_0^t \|u_{xxt}(s)\|^2 ds \leq C_4 + C_2\varepsilon^2\int_0^t \Big(\|u_{xtt}\|^2 + \|\theta_{xxt}\|^2\Big)(s)\, ds, \tag{6.5.30}$$

$$\|v_{xt}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 ds \leq C_4 + C_2\varepsilon^2\int_0^t \|v_{xtt}(s)\|^2 ds, \tag{6.5.31}$$

$$\|w_{xt}(t)\|^2 + \int_0^t \|w_{xxt}(s)\|^2 ds \leq C_4 + C_2\varepsilon^2\int_0^t \|w_{xtt}(s)\|^2 ds, \tag{6.5.32}$$

$$\|\theta_{xt}(t)\|^2 + \int_0^t \|\theta_{xxt}(s)\|^2 ds \leq C_4 + C_2\varepsilon^2\int_0^t \|\overrightarrow{\mathbf{v}}_{xxt}(s)\|^2 ds. \tag{6.5.33}$$

Proof. Similarly to lemma 2.5.2 (see also lemma 5.4.2 in Qin [99] or lemma 2.4 in Qin and Jiang [114]), we can obtain estimate (6.5.30)–(6.5.32).

Now we derive from (6.2.13) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tx}^2 dx =: \sum_{i=1}^6 D_i(t) \tag{6.5.34}$$

where

$$D_1(t) = -\int_0^L \left(\frac{r^2\kappa\theta_x}{\tau}\right)_{tx}\theta_{txx}\,dx, \qquad D_2(t) = -\int_0^L \left(\left(e_\tau + P - \delta\frac{(ru)_x}{\tau}\right)(ru)_x\right)_{xt}\theta_{tx}\,dx,$$

$$D_3(t) = \mu\int_0^L \left(\frac{(rv)_x^2 + r^2w_x^2}{x\tau}\right)_t \theta_{tx}\,dx, \qquad D_4(t) = -\int_0^L \left(e_{\theta tx}\theta_t + \frac{1}{2}e_{\theta t}\theta_{tx} + e_{\theta x}\theta_{tt}\right)\theta_{tx}\,dx,$$

$$D_5(t) = 2\mu\int_0^L (u^2+v^2)_{xt}\theta_{txx}\,dx, \qquad D_6(t) = \int_0^L (\tau(S_E)_R)_{tx}\theta_{tx}\,dx.$$

By virtue of theorems 6.2.1, 6.2.2, lemma 6.5.1 and (6.5.30)–(6.5.32), and using the embedding theorem and the Young inequality, we derive that for any $\varepsilon \in (0, 1)$,

$$D_1(t) \le -(2C_1)^{-1}\|\theta_{txx}\|^2 + C_2\left(\|u_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\tau_x\|^2\right), \tag{6.5.35}$$

$$D_2(t) \le \varepsilon^2\|u_{txx}\|^2 + C_2\varepsilon^{-2}\left(\|u_x\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|u_{xt}\|^2 + \|\tau_x\|_{H^1}^2\right), \tag{6.5.36}$$

$$\begin{aligned} D_3(t) \le \varepsilon^2\left(\|v_{txx}\|^2 + \|w_{txx}\|^2\right) &+ C_2\varepsilon^{-2}(\|v_x\|_{H^2}^2 + \|w_x\|_{H^2}^2 \\ &+ \|\theta_t\|_{H^1}^2 + \|\tau_x\|_{H^1}^2 + \|v_{xt}\|^2 + \|w_{xt}\|^2), \end{aligned} \tag{6.5.37}$$

$$D_4(t) \le \varepsilon^2\|\theta_{txx}\|^2 + C_2\varepsilon^{-2}\left(\|u_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\theta_x\|_{H^2}^2 + \|u_{xt}\|^2 + \|\tau_x\|^2\right), \tag{6.5.38}$$

$$D_5(t) \le \varepsilon^2\|\theta_{txx}\|^2 + C_2\varepsilon^{-2}\left(\|u_x\|_{H^1}^2 + \|v_x\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|v_t\|_{H^1}^2\right), \tag{6.5.39}$$

$$D_6(t) \le \varepsilon^2\|\theta_{xxt}\|^2 + C_1\varepsilon^{-2}\left\|(\tau(S_E)_R)_t\right\|^2. \tag{6.5.40}$$

Inserting (6.5.35)–(6.5.40) into (6.5.34), we obtain estimate (6.5.33). □

Subsequently, we establish some estimates in $\mathcal{H}^4$.

Lemma 6.5.3. *Under assumptions of theorem 6.2.3, we have for any $t > 0$,*

$$\begin{aligned} &\left\|\overrightarrow{\mathbf{v}}_{tt}(t)\right\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xt}(t)\right\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 \\ &\quad + \int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{ttx}\right\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xxt}\right\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{xxt}\|^2\right)(s)\,ds \le C_4, \end{aligned} \tag{6.5.41}$$

$$\begin{aligned} &\|\tau_{xxx}(t)\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}_{xxx}(t)\right\|_{H^1}^2 + \|\theta_{xxx}(t)\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}_{txx}(t)\right\|^2 + \|\theta_{txx}(t)\|^2 \\ &\quad + \int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{tt}\right\|^2 + \left\|\overrightarrow{\mathbf{v}}_{xxt}\right\|_{H^1}^2 + \|\theta_{tt}\|^2 + \|\theta_{xxt}\|_{H^1}^2\right)(s)\,ds \le C_4, \end{aligned} \tag{6.5.42}$$

$$\int_0^t \left(\|\tau_{xxx}\|_{H^1}^2 + \left\|\overrightarrow{\mathbf{v}}_{xxxx}\right\|_{H^1}^2 + \|\theta_{xxxx}\|_{H^1}^2\right)(s)\,ds \le C_4. \tag{6.5.43}$$

Proof. Adding up (6.5.3)–(6.5.5), we conclude

$$\begin{aligned}&\left\|\overrightarrow{\mathbf{v}}_{tt}(t)\right\|^2 + \|\theta_{tt}(t)\|^2 + \int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{xtt}\right\|^2 + \|\theta_{xtt}\|^2\right)(s)\,ds\\ &\quad\le C_4 + C_2\int_0^t \left(\left\|\overrightarrow{\mathbf{v}}_{xxt}\right\|^2 + \|\theta_{xxt}\|^2\right)(s)\,ds.\end{aligned} \tag{6.5.44}$$

Now multiplying (6.5.44) and (6.5.6) by ε and ε^2, respectively, then adding up the resultants and (6.5.30)–(6.5.33) and picking ε small sufficiently, we obtain (6.5.41).

Applying lemma 6.3.12, theorems 6.2.1, 6.2.2 and lemmas 6.5.1, 6.5.2 and using the Hölder inequality, we can derive from equation (6.2.20) that

$$\begin{aligned}\int_0^t \|\mathcal{I}_x\|^2\,ds &\le \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_x^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds\\ &= \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\left(\frac{r^{-2}\tau^2}{\omega^2}S^2 + \frac{\tau^2}{r^4}I^2\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds \le C_1,\end{aligned} \tag{6.5.45}$$

$$\begin{aligned}\int_0^t \|\mathcal{I}_{xx}\|^2\,ds &\le \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_{xx}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds\\ &\le C_1\int_0^t\int_0^L(\tau_x^2+\theta_x^2)\,dx\,ds + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_x^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds\\ &\quad + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega^2}S^2 + I^2\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds \le C_1,\end{aligned} \tag{6.5.46}$$

$$\begin{aligned}\int_0^t \|\mathcal{I}_{xxx}\|^2\,ds &\le \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_{xxx}^2\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds\\ &\le C_1\int_0^t\int_0^L(\tau_{xx}^2+\theta_{xx}^2+\tau_x^2+\theta_x^2)\,dx\,ds\\ &\quad + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}(I_{xx}^2+I_x^2+I^2)\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\,ds \le C_2.\end{aligned} \tag{6.5.47}$$

Similarly to lemma 2.5.3 or lemma 5.5.3 (see also lemma 5.4.5 in Qin [99]), we have the following estimates

$$\|\tau_{xxx}(t)\|^2 + \int_0^t \|\tau_{xxx}(s)\|^2\,ds \le C_4, \tag{6.5.48}$$

$$\big\|\overrightarrow{\mathbf{v}}_{xxx}(t)\big\|^2+\big\|\overrightarrow{\mathbf{v}}_{xx}(t)\big\|_{L^\infty}^2+\int_0^t\Big(\big\|\overrightarrow{\mathbf{v}}_{xxx}\big\|_{H^1}^2+\big\|\overrightarrow{\mathbf{v}}_{xx}\big\|_{W^{1,\infty}}^2\Big)(s)\,ds\le C_4. \tag{6.5.49}$$

Noting (6.4.14) and (6.5.10), using the embedding theorem, theorem 6.2.2 and lemmas 6.5.1, 6.5.2 and (6.5.45), (6.5.46), we can derive for any $t>0$,

$$\|\theta_{xxx}(t)\|^2+\|\theta_{xx}(t)\|_{L^\infty}^2+\int_0^t\Big(\|\theta_{xxx}\|_{H^1}^2+\|\theta_{xx}\|_{W^{1,\infty}}^2\Big)(s)\,ds\le C_4. \tag{6.5.50}$$

Similarly to lemma 2.5.3 or lemma 5.5.3, we also have

$$\big\|\overrightarrow{\mathbf{v}}_{xxxx}(t)\big\|^2+\int_0^t\Big(\big\|\overrightarrow{\mathbf{v}}_{txx}\big\|^2+\big\|\overrightarrow{\mathbf{v}}_{xxxx}\big\|^2\Big)(s)\,ds\le C_4. \tag{6.5.51}$$

Differentiating (6.2.13) with respect to t and using theorems 6.2.1 and 6.2.2, we can conclude

$$\|\theta_{txx}\|\le C_1(\|\theta_{tt}\|+\|\mathcal{I}_t\|)+C_2\Big(\big\|\overrightarrow{\mathbf{v}}_x\big\|_{H^1}+\|\tau_x\|+\|\theta_x\|_{H^2}+\|\theta_t\|_{H^1}+\big\|\overrightarrow{\mathbf{v}}_t\big\|_{H^1}\Big)\le C_4 \tag{6.5.52}$$

which, together with lemmas 6.5.1, 6.5.2 and (6.5.10), implies

$$\|\theta_{xxxx}(t)\|^2+\int_0^t\Big(\|\theta_{txx}\|^2+\|\theta_{xxxx}\|^2\Big)(s)\,ds\le C_4. \tag{6.5.53}$$

Therefore, combining (6.5.51) and (6.5.53), we get

$$\big\|\overrightarrow{\mathbf{v}}_{xxx}(t)\big\|_{L^\infty}^2+\|\theta_{xxx}(t)\|_{L^\infty}^2+\int_0^t\Big(\big\|\overrightarrow{\mathbf{v}}_{xxx}\big\|_{L^\infty}^2+\|\theta_{xxx}\|_{L^\infty}^2\Big)(s)\,ds\le C_4. \tag{6.5.54}$$

Similarly, noting (6.5.12) and (6.5.45), using the same method as in lemma 5.4.5 of Qin [99], we can obtain

$$\int_0^t\Big(\big\|\overrightarrow{\mathbf{v}}_{tt}\big\|^2+\|\theta_{tt}\|^2\Big)(s)\,ds\le C_4, \tag{6.5.55}$$

$$\|\tau_{xxxx}(t)\|^2+\int_0^t\|\tau_{xxxx}(s)\|^2\,ds\le C_4. \tag{6.5.56}$$

Differentiating (6.2.13) with respect to x and t, using theorem 6.2.2 and lemmas 6.5.1 and 6.5.2, we can derive

$$\|\theta_{txxx}\|\le C_1\|\theta_{ttx}\|+C_2\Big(\big\|\overrightarrow{\mathbf{v}}_x\big\|_{H^3}+\|\tau_x\|_{H^2}+\|\theta_x\|_{H^3}+\|\theta_{xt}\|+\big\|(\tau(S_E)_R)_{xt}\big\|\Big). \tag{6.5.57}$$

We know that

$$\begin{aligned}\int_0^t \left\|\tau(S_E)_R)_{xt}\right\|^2 ds \leq C_1 \int_0^t \Big(&\left\|(ru)_{xx}(S_E)_R\right\|^2 + \left\|(ru)_x((S_E)_R)_x\right\|^2 \\ &+ \left\|\tau_x((S_E)_R)_t\right\|^2 + \left\|((S_E)_R)_{xt}\right\|^2\Big) ds \\ =:& \sum_{i=1}^{4} B_i.\end{aligned} \tag{6.5.58}$$

Using equation (6.2.14) and the Sobolev embedding theorem, theorems 6.2.1, 6.2.2, (6.5.45) and (6.5.46), we have

$$\begin{aligned}B_1 &\leq C_1 \int_0^t \int_0^L (ru)_{xx}^2 \left(\int_0^{+\infty} \int_{S^2} S\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dxds \\ &\leq C_1 \int_0^t \int_0^L (ru)_{xx}^2 \left(\int_0^{+\infty} \int_{S^2} (rI)_x d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)^2 dxds \\ &\leq C_1 \int_0^t \int_0^L (ru)_{xx}^2 (\mathcal{I}_x^2 + \mathcal{I}^2)\, dxds \\ &\leq C_1 \int_0^t \left\|(ru)_{xx}\right\|^2 \Big(\|\mathcal{I}_x\|_{L^\infty}^2 + \|\mathcal{I}\|_{L^\infty}^2\Big) ds \\ &\leq C_1 \int_0^t \Big(\|\mathcal{I}_{xx}\|^2 + \|\mathcal{I}_x\|^2\Big) \leq C_1.\end{aligned} \tag{6.5.59}$$

Similarly, we also derive from (6.5.45)–(6.5.47) that

$$B_2 \leq C_1 \int_0^t \Big(\|\mathcal{I}_{xxx}\|^2 + \|\mathcal{I}_{xx}\|^2 + \|\mathcal{I}_x\|^2\Big) \leq C_2. \tag{6.5.60}$$

Using lemma 6.3.13 and theorems 6.2.1, 6.2.2, we can derive

$$\begin{aligned}B_3 &\leq C_1 \int_0^t \int_0^L \tau_x^2 \left(\int_0^{+\infty} \int_{S^2} S\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\right)_t^2 dxds \leq C_1 \int_0^t \int_0^L \tau_x^2 \int_0^{+\infty} \int_{S^2} (\sigma_a(B-I))_t^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon dxds \\ &\leq C_1 \int_0^t \int_0^L (\tau_x^2 + (ru)_x^2\tau_x^2 + \theta_t^2\tau_x^2)\, dxds + C_1 \int_0^t \int_0^L \int_0^{+\infty} \int_{S^2} I_t^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon dxds \\ &\leq C_1 \int_0^t \Big(\|u_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\tau_x\|^2\Big) ds + C_1 \int_0^t \int_0^L \int_0^{+\infty} \int_{S^2} I_t^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon dxds \leq C_1.\end{aligned} \tag{6.5.61}$$

In the same manner, we also get

$$B_4 \leq C_1 \int_0^t \Big(\|u_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\tau_x\|_{H^1}^2\Big) ds + C_1 \int_0^t \int_0^L \int_0^{+\infty} \int_{S^2} I_{xt}^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon dxds. \tag{6.5.62}$$

Meanwhile, it follows from (6.2.14) and lemma 6.3.12 that

$$\begin{aligned}
&\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_{xt}^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\, dx ds\\
&\quad= \int_0^t\int_0^L\int_0^{+\infty}\int_{S^2}\left(\frac{r^{-1}\tau}{\omega}S+\frac{\tau}{r^2}I\right)_t^2 d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\, dx ds\\
&\quad\le C_1 + C_1\int_0^t\int_0^L (u_x^2+\theta_t^2)\, dx ds + C_1\int_0^t\int_0^L\int_0^{+\infty}\int_{S^2} I_t^2\, d\overrightarrow{\boldsymbol{\Omega}}\, d\epsilon\, dx ds\\
&\quad\le C_1.
\end{aligned} \tag{6.5.63}$$

Thus we derive from (6.5.57)–(6.5.63) that

$$\int_0^t \|\theta_{txxx}(s)\|^2 ds \le C_4. \tag{6.5.64}$$

Differentiating (6.2.10) with respect to x three times, using lemmas 6.5.1, 6.5.2 and theorem 6.2.2 and applying the Poincaré inequality, we derive

$$\|u_{xxxxx}\| \le C_1\|u_{txxx}\| + C_2\big(\|u_x\|_{H^3}+\|\tau_x\|_{H^3}+\|\theta_x\|_{H^3}+\|v_x\|_{H^2}\big). \tag{6.5.65}$$

Thus we derive

$$\int_0^t \|u_{xxxxx}(s)\|^2 ds \le C_4. \tag{6.5.66}$$

Using the same technique, we can derive from (6.2.11)–(6.2.13) that

$$\|v_{xxxxx}\| \le C_1\|v_{txxx}\| + C_2\big(\|v_x\|_{H^3}+\|\tau_x\|_{H^3}+\|u_x\|_{H^2}\big), \tag{6.5.67}$$

$$\|w_{xxxxx}\| \le C_1\|w_{txxx}\| + C_2\big(\|w_x\|_{H^3}+\|\tau_x\|_{H^3}\big), \tag{6.5.68}$$

$$\|\theta_{xxxxx}\| \le C_1(\|\theta_{txxx}\|+\|\mathcal{I}_{xxx}\|) + C_2\Big(\big\|\overrightarrow{\mathbf{v}}_x\big\|_{H^3}+\|\tau_x\|_{H^3}+\|\theta_x\|_{H^3}\Big). \tag{6.5.69}$$

Therefore, we have

$$\int_0^t \|\theta_{xxxxx}(s)\|^2 ds \le C_4. \tag{6.5.70}$$

Hence, we complete the proofs of (6.5.56) and (6.5.57). □

Finally, we concern the estimate on $\mathcal{I}$.

Lemma 6.5.4. *Under assumptions in theorem 6.2.3, the following estimate holds for any* $t>0$,

$$\|\mathcal{I}(t)\|_{H^5} \le C_4. \tag{6.5.71}$$

Proof. By equation (6.2.14), we have

$$\begin{aligned}\|\mathcal{I}_{xxxx}(t)\|^2 &= \int_0^L \left(\int_0^{+\infty}\int_{S^2}\frac{1}{\omega}(r^{-1}\tau S)_{xxx}d\overrightarrow{\Omega}\,d\epsilon + \int_0^{+\infty}\int_{S^2}(r^{-2}\tau I)_{xxx}d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\le C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^4+\tau_x+\tau_x^2+\tau_{xx}+\tau_{xxx})\left(\frac{1}{\omega}S+I\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^3+\tau_x+\tau_{xx})\left(\frac{1}{\omega}S_x+I_x\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^2+\tau_x)\left(\frac{1}{\omega}S_{xx}+I_{xx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xxx}+I_{xxx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx =: \sum_{i=1}^4 F_i. \end{aligned}\tag{6.5.72}$$

Now we estimate each term in (6.5.72). Noting the expressions of S and $\mathcal{I}$, we have

$$\begin{aligned}F_1 &\le C_1\int_0^L(\tau^8+\tau_x^2+\tau_x^4+\tau_{xx}^2+\tau_{xxx}^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S+I\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\le C_1\int_0^L(\tau^2+\tau_x^2+\tau_x^4+\tau_{xx}^2+\tau_{xxx}^2)\left(C_1+\|\mathcal{I}\|^2\right)dx \le C_1\|\tau_{xxx}\|^2\le C_4.\end{aligned}\tag{6.5.73}$$

Noting assumptions on σ_a and σ_s, applying the Sobolev embedding theorem and lemma 6.3.13, we have

$$\begin{aligned}F_2 &\le C_1\int_0^L(\tau^6+\tau_x^2+\tau_{xx}^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_x+I_x\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\le C_1\int_0^L(\tau^2+\tau_x^2+\tau_{xx}^2)\left(\tau_x^2+\theta_x^2+\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_x d\overrightarrow{\Omega}\,d\epsilon\right)^2\right)dx\\
&\le C_1\int_0^L(\tau^2+\tau_x^2+\tau_{xx}^2)(\tau_x^2+\theta_x^2+\mathcal{I}_x^2)dx\\
&\le C_1\left(\|\tau_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\mathcal{I}_x\|_{H^1}^2\right)\le C_4.\end{aligned}\tag{6.5.74}$$

Similarly, we can also derive from lemmas 6.3.13 and 6.4.3 that

$$\begin{aligned}F_3 &\le C_1\int_0^L(\tau^4+\tau_x^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xx}+I_{xx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\
&\le C_1\int_0^L(\tau^4+\tau_x^2)\Bigg(\tau_{xx}^2+\theta_{xx}^2+\tau_x^2+\theta_x^2+\tau_x^2\theta_x^2\\
&\qquad+\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_{xx}d\overrightarrow{\Omega}\,d\epsilon\right)^2\Bigg)dx\\
&\le C_1\int_0^L(\tau^4+\tau_x^2)\left(\tau_{xx}^2+\theta_{xx}^2+\tau_x^2+\theta_x^2+\tau_x^2\theta_x^2+\|\mathcal{I}_{xx}\|^2\right)dx\\
&\le C_1\left(\|\tau_x\|_{H^1}^2+\|\theta_x\|_{H^1}^2+\|\mathcal{I}_x\|_{H^1}^2\right)\le C_4,\end{aligned}\tag{6.5.75}$$

$$\begin{aligned}F_4 &\le C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xxx}+I_{xxx}\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\le C_1\Big(\|\tau_x\|_{H^2}^2+\|\theta_x\|_{H^2}^2\Big)+C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_{xxx}\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\le C_1\Big(\|\tau_x\|_{H^2}^2+\|\theta_x\|_{H^2}^2+\|\mathcal{I}_x\|_{H^2}^2\Big)\le C_4.\end{aligned}\tag{6.5.76}$$

Inserting (6.5.73)–(6.5.76) into (6.5.72), we have

$$\|\mathcal{I}_{xxxx}(t)\|\le C_4.\tag{6.5.77}$$

Analogously, we also derive from equation (6.2.14) that

$$\begin{aligned}\|\mathcal{I}_{xxxxx}(t)\|^2 &= \int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}(r^{-1}\tau S)_{xxxx}+(r^{-2}\tau I)_{xxxx}\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\le C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^5+\tau_x+\tau_x^2+\tau_{xx}+\tau_x\tau_{xx}+\tau_{xxx}+\tau_{xxxx})\left(\frac{1}{\omega}S+I\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^4+\tau_x+\tau_x^2+\tau_{xx}+\tau_{xxx})\left(\frac{1}{\omega}S_x+I_x\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^3+\tau_x+\tau_{xx})\left(\frac{1}{\omega}S_{xx}+I_{xx}\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}(\tau^2+\tau_x)\left(\frac{1}{\omega}S_{xxx}+I_{xxx}\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\quad+\int_0^L\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xxxx}+I_{xxxx}\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx=:\sum_{i=1}^5G_i.\end{aligned}\tag{6.5.78}$$

Using the same technique as in (6.5.73)–(6.5.76), we can also get the following estimates

$$\begin{aligned}G_1&\le C_1\int_0^L(\tau^{10}+\tau_x^2+\tau_x^4+\tau_{xx}^2+\tau_{xxx}^2+\tau_x^2\tau_{xx}^2+\tau_{xxxx}^2)\\&\qquad\times\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega^2}S^2+I^2\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\,dx\\&\le C_1\|\tau_{xxxx}\|^2\le C_4,\end{aligned}\tag{6.5.79}$$

$$\begin{aligned}G_2&\le C_1\int_0^L(\tau^8+\tau_x^2+\tau_x^4+\tau_{xx}^2+\tau_{xxx}^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_x+I_x\right)d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2dx\\&\le C_1\int_0^L(\tau^8+\tau_x^2+\tau_x^4+\tau_{xx}^2+\tau_{xxx}^2)\left(\tau_x^2+\theta_x^2+\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_x\,d\overrightarrow{\mathbf{\Omega}}\,d\epsilon\right)^2\right)dx\\&\le C_1\Big(\|\tau_x\|_{H^3}^2+\|\theta_x\|^2+\|\mathcal{I}_x\|_{H^1}^2\Big)\le C_4,\end{aligned}\tag{6.5.80}$$

$$\begin{aligned}G_3 &\le C_1 \int_0^L (\tau^6+\tau_x^2+\tau_{xx}^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xx}+I_{xx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\ &\le C_1 \int_0^L (\tau^6+\tau_x^2+\tau_{xx}^2)\left(\tau_{xx}^2+\theta_{xx}^2+\tau_x^2+\theta_x^2+\tau_x^2\theta_x^2+\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_{xx}d\overrightarrow{\Omega}\,d\epsilon\right)^2\right)dx\\ &\le C_1\Big(\|\tau_x\|_{H^2}^2+\|\theta_x\|_{H^1}^2+\|\mathcal{I}_x\|_{H^1}^2\Big)\le C_4, \qquad (6.5.81)\end{aligned}$$

$$\begin{aligned}G_4 &\le C_1 \int_0^L (\tau^4+\tau_x^2)\left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xxx}+I_{xxx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\ &\le C_1\Big(\|\tau_{xx}\|_{H^1}^2+\|\theta_{xx}\|_{H^1}^2\Big)+C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_{xxx}d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\ &\le C_1\Big(\|\tau_{xx}\|_{H^1}^2+\|\theta_{xx}\|_{H^1}^2+\|\mathcal{I}_x\|_{H^2}^2\Big)\le C_4. \qquad (6.5.82)\end{aligned}$$

Using the Sobolev embedding theorem and (6.5.77), we have

$$\begin{aligned}G_5 &\le C_1 \int_0^L \left(\int_0^{+\infty}\int_{S^2}\left(\frac{1}{\omega}S_{xxxx}+I_{xxxx}\right)d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\ &\le C_1\Big(\|\tau_{xx}\|_{H^2}^2+\|\theta_{xx}\|_{H^2}^2\Big)+C_1\int_0^L\left(\int_0^{+\infty}\int_{S^2}(1+\sigma_a+\sigma_s)I_{xxxx}d\overrightarrow{\Omega}\,d\epsilon\right)^2 dx\\ &\le C_1\Big(\|\tau_{xx}\|_{H^2}^2+\|\theta_{xx}\|_{H^2}^2+\|\mathcal{I}_x\|_{H^3}^2\Big)\le C_4. \qquad (6.5.83)\end{aligned}$$

Inserting (6.5.79)–(6.5.83) into (6.5.78), we get

$$\|\mathcal{I}_{xxxxx}(t)\|\le C_4 \qquad (6.5.84)$$

which, along with (6.5.77), leads to (6.5.71). □

By lemmas 6.5.1–6.5.4, we have proved the global existence of solutions in $\mathcal{H}^4$ to the problem (6.2.9)–(6.2.17) with arbitrary initial datum $(\tau_0, \overrightarrow{\mathbf{v}}_0, \theta_0, \mathcal{I}_0)\in\mathcal{H}^4$ and the uniqueness of solutions follows from that of solutions in $\mathcal{H}^1$ or in $\mathcal{H}^2$.

6.5.2 Asymptotic Behavior in $\mathcal{H}^4$

In this subsection, we shall show the asymptotic behavior of global solutions in $\mathcal{H}^4$.

Lemma 6.5.5. *Under assumptions in theorem 6.2.3, we have, as* $t\to+\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^4}\to 0,\quad \big\|\overrightarrow{\mathbf{v}}(t)\big\|_{H^4}\to 0,\quad \big\|\theta(t)-\overline{\theta}\big\|_{H^4}\to 0. \qquad (6.5.85)$$

Proof. Differentiating (6.2.9) with respect to x three times and then multiplying the resultant by τ_{xxx} in $L^2(0,\, L)$, we have

$$\frac{d}{dt}\|\tau_{xxx}\|^2 \leq C_2\Big(1+\|u_{xxx}\|^2+\|u_{xxxx}\|^2+\|\tau_{xxx}\|^2\Big) \tag{6.5.86}$$

which, along with (6.5.51) and theorems 6.2.1, 6.2.2 and 1.2.10, leads to, as $t \to +\infty$,

$$\|\tau_{xxx}(t)\| \to 0. \tag{6.5.87}$$

Similarly, differentiating (6.2.9) with respect to x four times, we can obtain

$$\frac{d}{dt}\|\tau_{xxxx}\|^2 \leq C_2\Big(1+\|u_{xxx}\|^2+\|u_{xxxx}\|^2+\|u_{xxxxx}\|^2+\|\tau_{xxxx}\|^2\Big) \tag{6.5.88}$$

which, along with (6.5.56) and theorems 6.2.1, 6.2.2 and 1.2.10, yields, as $t \to +\infty$,

$$\|\tau_{xxxx}(t)\| \to 0, \tag{6.5.89}$$

which, combined with the Poincaré inequality, gives, as $t \to +\infty$,

$$\|\tau(t)-\overline{\tau}\|_{H^4} \to 0. \tag{6.5.90}$$

From lemma 6.5.2, we can easily get

$$\begin{aligned}
&\frac{d}{dt}\|\theta_{xt}(t)\|^2 + C_1^{-1}\|\theta_{xxt}(t)\|^2 \\
&\quad \leq \varepsilon\Big(\|\theta_{xxxt}(t)\|^2+\big\|\overrightarrow{\mathbf{v}}_{xxt}(t)\big\|^2\Big) + C_2\big(\big\|\overrightarrow{\mathbf{v}}(t)\big\|_{H^1}^2+\|\theta_x(t)\|_{H^2}^2 \\
&\qquad + \|\theta_{xt}(t)\|^2+\|\tau_x(t)\|_{H^2}^2+\big\|(\tau(S_E)_R)_{xt}(t)\big\|^2\big).
\end{aligned} \tag{6.5.91}$$

From the calculation of (6.5.59)–(6.5.62), we have

$$\big\|(\tau(S_E)_R)_{xt}(t)\big\|^2 \leq C_1\Big(\|\mathcal{I}(t)\|_{H^3}^2+\|\mathcal{I}_{xt}(t)\|^2+\|u(t)\|_{H^1}^2+\|\theta_t(t)\|_{H^1}^2+\|\theta_x(t)\|_{H^1}^2+\|\tau_{xx}(t)\|^2\Big) \tag{6.5.92}$$

which, combined with theorem 1.2.10 and (6.5.91), leads to, as $t \to +\infty$,

$$\|\theta_{xt}(t)\| \to 0. \tag{6.5.93}$$

Noting that, as $t \to +\infty$,

$$\|\mathcal{I}(t)\|_{H^1} \to 0 \tag{6.5.94}$$

and using (6.4.14), we have, as $t \to +\infty$,

$$\|\theta_{xxx}(t)\| \to 0. \tag{6.5.95}$$

From lemma 6.5.1, we can easily obtain, for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{d}{dt}\|\theta_{tt}(t)\|^2 + C_1^{-1}\|\theta_{xxt}(t)\|^2\\&\quad\le \varepsilon\Big(\|\theta_{xtt}(t)\|^2 + \big\|\overrightarrow{\mathbf{v}}_{xtt}(t)\big\|^2\Big) + C_2\big(\big\|\overrightarrow{\mathbf{v}}_{xx}(t)\big\|^2 + \big\|\overrightarrow{\mathbf{v}}_{xt}(t)\big\|^2\\&\qquad + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \big\|(\tau(S_E)_R)_{tt}(t)\big\|^2\big).\end{aligned} \tag{6.5.96}$$

From equation (6.2.14), we have

$$\big\|(\tau(S_E)_R)_{tt}(t)\big\|^2 \le C_1\Big(\|u_t(t)\|_{H^1}^2 + \|\theta_{tt}(t)\|^2 + \|\mathcal{I}_{tt}(t)\|^2 + \|\mathcal{I}_t(t)\|^2 + \|\mathcal{I}(t)\|^2\Big) \tag{6.5.97}$$

which, by theorem 1.2.10 and (6.5.96), leads to, as $t \to +\infty$,

$$\|\theta_{tt}(t)\| \to 0. \tag{6.5.98}$$

On the other hand, we have

$$\|\theta_{xxt}(t)\| \le C_2\big(\|\theta_{tt}(t)\| + \|\tau_x(t)\| + \big\|\overrightarrow{\mathbf{v}}_x(t)\big\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|\theta_{xxx}(t)\| + \|\mathcal{I}_{xx}(t)\|\big),$$

which, combined with lemma 6.4.4 and (6.5.93)–(6.5.97), yields, as $t \to +\infty$,

$$\|\theta_{xxt}(t)\| \to 0. \tag{6.5.99}$$

Thus it follows from (6.5.98) and (6.5.10) that, as $t \to +\infty$,

$$\|\theta_{xxxx}(t)\| \to 0, \tag{6.5.100}$$

which, obviously, along with theorem 1.2.10 and the Poincaré inequality, leads to our desired estimate $\big\|\theta(t) - \overline{\theta}\big\|_{H^4} \to 0$, as $t \to +\infty$. Analogously, applying lemmas 6.5.1, 6.5.2 and theorem 1.2.10, we can derive from (6.2.10)–(6.2.12) that $\big\|\overrightarrow{\mathbf{v}}(t)\big\|_{H^4} \to 0$. Here we omit it. □

Lemma 6.5.6. *Under assumptions in theorem 6.2.3, we have, as $t \to +\infty$,*

$$\|\mathcal{I}(t)\|_{H^5} \to 0. \tag{6.5.101}$$

Proof. We can derive from (6.5.73)–(6.5.76) that

$$F_1 \le C_1\|\tau_{xxx}\|^2, \tag{6.5.102}$$

$$F_2 \le C_1\Big(\|\tau_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\mathcal{I}_x\|_{H^1}^2\Big), \tag{6.5.103}$$

$$F_3 \le C_1\Big(\|\tau_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\mathcal{I}_{xx}\|^2\Big), \tag{6.5.104}$$

$$F_4 \leq C_1 \Big(\|\tau_x\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 + \|\mathcal{I}_x\|^2 + \|\mathcal{I}_{xx}\|^2 \Big). \tag{6.5.105}$$

Applying (6.5.72), (6.5.102)–(6.5.105) and theorems 6.2.1, 6.2.2, we have, as $t \to +\infty$,

$$\|\mathcal{I}_{xxxx}(t)\|^2 \to 0. \tag{6.5.106}$$

Similarly, using (6.5.79)–(6.5.83), (6.5.106) and theorems 6.2.1, 6.2.2, we also have, as $t \to +\infty$,

$$\|\mathcal{I}_{xxxxx}(t)\|^2 \to 0. \tag{6.5.107}$$

Therefore, we complete the proof. □

Till now we have completed the proof of theorem 6.2.3. □

6.6 Bibliographic Comments

About the infrarelativistic model with radiation, we have recalled some known results in section 6.5. For the cylindrically symmetric Navier–Stokes equations, we would like to refer to [99, 103, 114] and the references therein. However, it should be pointed out that the authors have established main results in [99, 103, 114] only when the initial total energy is sufficiently small. In this chapter, our results have improved these known results on the cylindrically symmetric case.

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